Stability Analysis Using Dipole Oscillation Model For RHIC with E-Cooling

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Section 1. introduction

In 1998, a substantial shorter beam life time was observed as soon as the E-Cooler was turned on in Celsius and this phenomenon has been called 'electron heating'. Similar phenomena have also been observed by other facilities such as NAP-M, Fermi lab, Indiana, TARN II and COSY. Although a nonlinear electric field is regarded as an important reason for the fast beam loss in Celsius due to the fact that the electron beams has a smaller radius than the ion beam, the coherent ion-electron beam interaction may also play a role. For RHIC e-cooler, since the electron beam and the ion beam have essentially the same beam size, the nonlinear electric field effects are greatly reduced and the coherent ion-electron interaction could be important for the ion beam stability. V.V.Parkhamchuk and V.B.Reva developed a dipole oscillation model to estimate the growth rate due to transversal coherent oscillation induced by electron beam [1, 2]. It is also shown that this coherent effect could be amplified in the presence of the ion clouds ionized from the residue gas[3]. This model is reviewed and applied to the RHIC electron cooling parameters. In section 2, the longitudinal two stream coupling is studied and the instability threshold is shown for the designed RHIC parameters. In section 3, the transverse two stream coupling equation is solved and the growth rate of the transverse coherent oscillation is estimated for the magnetized electron cooling scheme. The effects of the ion clouds in the cooling section have been taken into account and the dependence of the growth rate on the neutralization factor is derived. The stability analysis of the ion clouds motion inside the cooling section has also been made in order to estimate the neutralization ratio. It is shown that, in the presence of a strong longitudinal magnetic field, the ion clouds may not be removed by simply making a gap due to the Larmor oscillation resonance. The calculation for non-magnetized electron cooling design is given in subsection 3.4 and it shows that the designed electron density is three orders of magnitude smaller than the transverse instability threshold.

Section 2. Longitudinal-Longitudinal Coupling

2.1Langmuir oscillation equations of motion

In the presence of the electron beam, the longitudinal electrostatic oscillations (Langmuir oscillations) can be excited and amplified from turn to turn, leading to an ion beam instability. As shown in Fig.1, the electron and ion displacement from their equilibrium position make the local longitudinal boundaries carry opposite surface charge

$$\begin{aligned}\sigma(z,t) &= \sigma_e(z,t) + \sigma_i(z,t) \\ &= -en_e s_e(z,t) + Z_i e n_i s_i(z,t)\end{aligned}$$

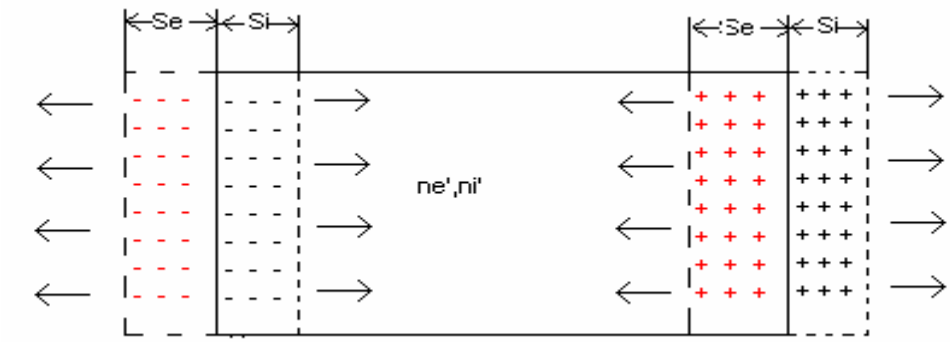

Fig.1.The coherent displacement of all ions/electrons within a local region of the ion beam, Si, makes the charge density inside the region different from the equilibrium and thus induces local electrostatic field, which in turn act back on the perturbing particles and make them oscillate around the equilibrium position.

where Z_i is the ion charge number. Assuming the volume charge density variation due to the displacement within the considered region is negligible, the electrostatic field due to the displacement is

$$E_z(z,t) = \frac{1}{\varepsilon_0}\left(en_e s_e(z,t) - Z_i e n_i s_i(z,t)\right) \tag{2.1}$$

The negative sign comes from the fact that the positive longitudinal displacement of positive charge particles introduces positive surface charge to the right boundary and thus creates a negative electrostatic field. The factor of 2 comes from the fact that both boundaries contribute the same amount of surface charge with opposite sign. The equations of longitudinal motion for an electron/ion within the considered region are thus,

$$m_e \frac{d^2 z_e}{dt^2} = -\frac{n_e e^2}{\varepsilon_0} s_e(z,t) + \frac{Z_i n_i e^2}{\varepsilon_0} s_i(z,t) \tag{2.2}$$

$$M_i \frac{d^2 z_i}{dt^2} = \frac{Z_i n_e e^2}{\varepsilon_0} s_e(z,t) - \frac{Z_i^2 n_i e^2}{\varepsilon_0} s_i(z,t) \tag{2.3}$$

Since inside the considered region, the longitudinal position of each particle is the sum of its equilibrium position and the longitudinal displacement, i.e.

$$z_{e,i} = z_{c,i}^{0} + s_{e,i}$$

and since the unperturbed equilibrium position for each particle is independent of time, the equation of motion for the displacements have been obtained as following,

$$\frac{d^2 s_e}{dt^2} + \omega_{pe}^2 s_e = \omega_{ei}^2 s_i \tag{2.4}$$

$$\frac{d^2 s_i}{dt^2} + \omega_{pi}^2 s_i = \omega_{ie}^2 s_e \tag{2.5}$$

where the plasma frequencies are defined as[1]

$$\omega_{pe} = \sqrt{\frac{n_e e^2}{m_e \varepsilon_0}} = 1.504 \times 10^9 s^{-1}$$

$$\omega_{pi} = \sqrt{\frac{Z_i^2 n_i e^2}{M_i \varepsilon_0}} = 6.518 \times 10^6 s^{-1}$$

$$\omega_{ie} = \sqrt{\frac{Z_i n_e e^2}{M_i \varepsilon_0}} = 2.23 \times 10^7 s^{-1}$$

$$\omega_{ei} = \sqrt{\frac{Z_i n_i e^2}{m_e \varepsilon_0}} = 4.396 \times 10^8 s^{-1}$$

2.2 Transfer matrix for Langmuir oscillation in cooling section.

Given the initial condition, the equation (2.4) and (2.5) can be solved and thus the displacements at the back end of the cooling section can be obtained. Since the electron beam is much colder than the ion beam and will be renewed for each turn, the initial condition for the electron beam can be set to

$$s_e(z,0) = 0 \tag{2.6}$$

$$\frac{d}{dt} s_e(z,0) = 0 \tag{2.7}$$

where $t = 0$ corresponds to front end of the cooling section.

Equation (2.1) can be rewritten as,

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[1] All the numbers are given for the commoving frame densities $n_i = 7.697 \times 10^{11} m^{-3}$ and $n_e = 7.117 \times 10^{16} m^{-3}$, which correspond to RHIC magnetized cooling parameters.

$$E_z(z,t) = \frac{m_e}{e}\left(\omega_{pe}^2 s_e(z,t) - \omega_{ei}^2 s_i(z,t)\right) \tag{2.8}$$

By applying equation (2.4), (2.5) and (2.8), the differential equation for the longitudinal electric field can be derived as

$$\frac{d^2}{dt^2}E_z(z,t) = -\omega_0^2 E_z(z,t)$$

where

$$\omega_0 = \sqrt{\omega_{pe}^2 + \omega_{pi}^2} = 1.504\times10^9\, s^{-1}.$$

Thus the longitudinal electric field due to the displacements of ions and electrons in the considered region turns out to be

$$E_z(z,t) = \hat{E}_z(z)\cos(\omega_0 t + \varphi) \tag{2.10}$$

At $t = 0$, by equating equation (2.8) and (2.10), one gets

$$\cos(\varphi) = -\frac{m_e}{\hat{E}_z(z)e}\omega_{ei}^2 s_i(z,0) \tag{2.11}$$

$$\sin(\varphi) = \frac{m_e}{\hat{E}_z(z)\omega_0 e}\omega_{ei}^2 \dot{s}_i(z,0) \tag{2.12}$$

By equation (2.3) and (2.10), the equation of motion for ion displacement can be rewritten to

$$\frac{d^2}{dt^2}s_i(z,t) = \frac{Z_i e}{M_i}\hat{E}_z(z)\cos(\omega_0 t + \varphi) \tag{2.13}$$

Integrating equation (2.13) over t, the velocity of the particles' longitudinal shift can be obtained,

$$\frac{d}{dt}s_i(z,t) = -\xi\omega_0 \sin(\omega_0\tau)s_i(z,0) + \left[\xi(\cos(\omega_0\tau) - 1) + 1\right]\dot{s}_i(z,0) \tag{2.14}$$

where $\xi = \frac{\omega_{pi}^2}{\omega_0^2} = 1.879\times10^{-5}$, $\tau = \frac{l_{cool}}{\gamma c}$ is the flight time in the commoving frame and l_{cool} is the length of the cooling section. Then we can get the solution of the displacement, s_i by integrating equation (2.14),

$$s_i(z,t) = \left[\xi(\cos(\omega_0\tau) - 1) + 1\right]s_i(z,0) + \frac{1}{\omega_0}\left[\xi\sin(\omega_0\tau) + (1-\xi)\omega_0\tau\right]\dot{s}_i(z,0) \tag{2.15}$$

From equation (2.14) and (2.15), the transfer matrix for the ion displacement due to coherent Langmuir oscillation in the cooling section is

$$M_{langmuir} = \begin{pmatrix} \xi(\cos(\omega_0\tau)-1)+1 & \frac{1}{\omega_0}\left[\xi\sin(\omega_0\tau)+(1-\xi)\omega_0\tau\right] \\ -\xi\omega_0\sin(\omega_0\tau) & \xi(\cos(\omega_0\tau)-1)+1 \end{pmatrix} \tag{2.16}$$

Thus for each turn passing through the cooling section, the ions' local longitudinal displacements varies as

$$\begin{pmatrix} s_i \\ \dot{s}_i \end{pmatrix}_\tau = M_{langmuir}\begin{pmatrix} s_i \\ \dot{s}_i \end{pmatrix}$$

2.3 Determinant of the transfer matrix

The determinant of the transfer matrix can be represented in terms of the plasma frequencies as following,

$$\left|M_{langmuir}\right| = 1 + \frac{\omega_{ie}^2\omega_{ei}^2}{\omega_0^4}\omega_0\tau\sin(\omega_0\tau) + \frac{2\omega_{ie}^2\omega_{ei}^2}{\omega_0^4}(\cos(\omega_0\tau)-1) \tag{2.17}$$

where the relation $\omega_{pi}^2\omega_{pe}^2 = \omega_{ei}^2\omega_{ie}^2$ is used in the derivation. If $\left|M_{langmuir}\right| > 1$, the electron beam will transfer energy to the ion oscillation and thus increase the local electrostatic oscillation and cause instability. From equation (2.17), the condition for $\left|M_{langmuir}\right| > 1$ can be depicted as follows,

$$\left|M_{langmuir}\right| - 1 = \frac{4\omega_{ie}^2\omega_{ei}^2}{\omega_0^4}\sin^2\left(\frac{\omega_0\tau}{2}\right)\left[\frac{\omega_0\tau}{2}\cot\left(\frac{\omega_0\tau}{2}\right)-1\right] \tag{2.18}$$

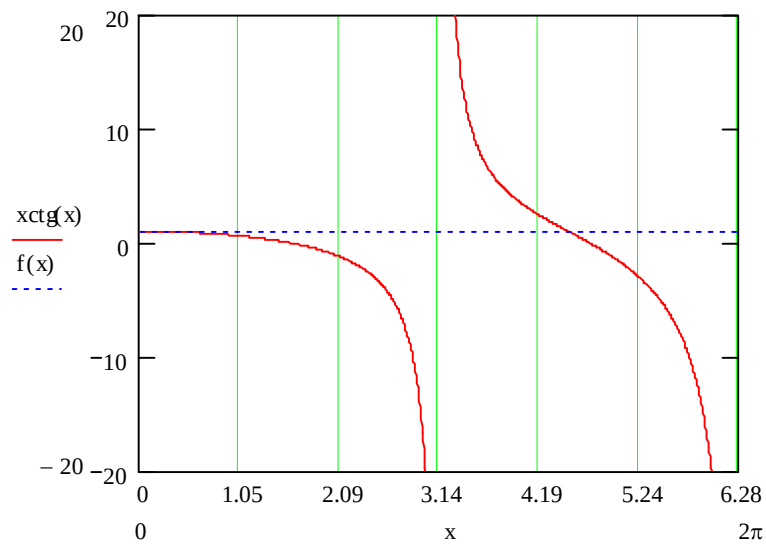


Fig.2 Plot to show the sign of the $\left|M_{langmuir}\right|-1$. The red solid curve is for $y(x)=x\cot(x)$ and the blue dot line is for $y(x)=1$. The plot shows the sign changes at $x=\pi$, i.e. $\omega_0\tau=2\pi$.

Thus the threshold for the determinant of the transfer matrix to be bigger than 1 is

$$\omega_0\tau=2\pi \tag{2.19}$$

The RHIC gold beam parameters in the cooling section are shown in Table 1

$\varepsilon_{nx},\varepsilon_{ny}$	15 π mm.mrad.
β_x,β_y	60 meter
N_i (Particles per bunch)	10^9
l_i (rms Bunch length)	0.37 meter
γ (Beam energy)	100
σ_x,σ_y (Ion beam size)	1.2 mm
n_i (Ion beam density in beam frame)	$7.697\times10^{11}\,meter^{-3}$
τ (Cooling section flight time in beam frame)	2×10^{-9} s

Table.1 RHIC gold ion beam parameters in the cooling section. For simplicity, a round beam approximation is used in the calculation. The emittance refers to 95% emittance.

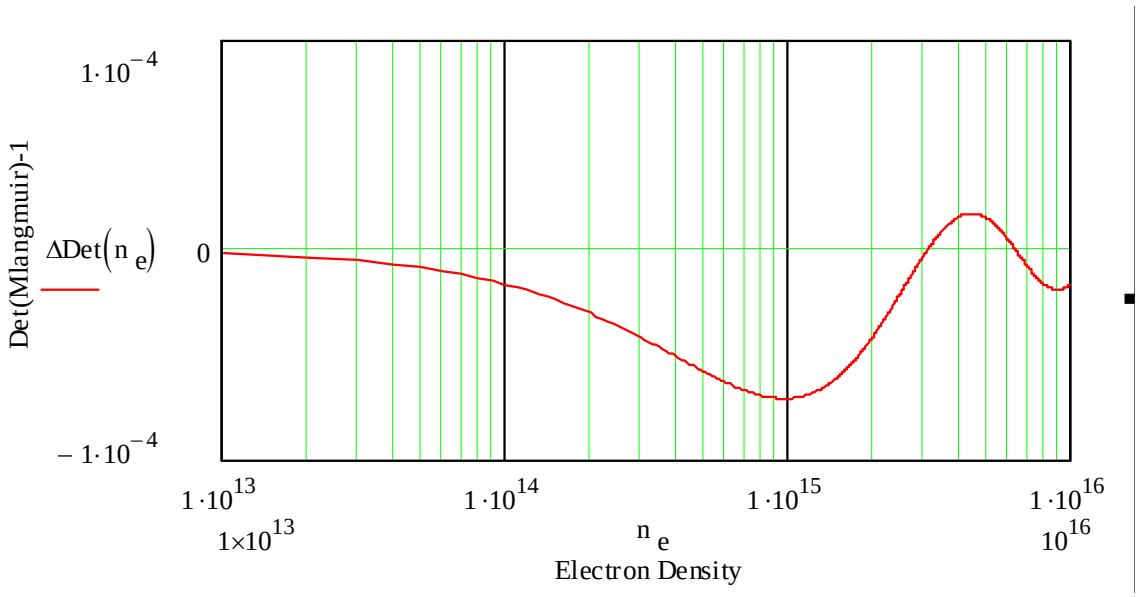


Fig.3 $Det(M_{langmuir})$ dependence of the electron beam density

As shown in Fig.3, the longitudinal electrostatic oscillation puts an electron density limitation, $n_{e,th} = 3.102 \times 10^{15} m^{-3}$, which correspond to $\omega_0 \tau = 2\pi$, for the ion beam to be stable. For the current electron cooler design, the electron beam has the parameters shown in Table 2,

Q_e (Electron charge per bunch)	20 nC
N_e (Electron number per bunch)	1.25×10^{11}
σ_e (Electron rms beam size)	$1.225 \times 10^{-3} m$

Table.2 Electron beam parameters for the current electron cooler design.

Thus, the density limitation corresponds to a bunch length limitation of the electron beam,

$$l_e \geq \frac{N_e}{\pi \sigma_e^2 n_{e,th} \gamma} = 1.147 \times 10^{-2} m$$

There are other limitations on the electron beam bunch length set by the requirement of optimizing the cooling force. For example if the electron bunch is shorter than 18 cm, Debye screening starts to reduce the cooling force. Since the electron beam bunch length is already 2 cm at the exit of the gun and stretchers have been designed to stretch the beam for higher cooling rate, this coherent longitudinal instability does not affect the current RHIC magnetized electron cooler design.

2.4 Eigenvalues of the transfer matrix

Although $\left|M_{langmuir}\right| \le 1$ is necessary condition for the ion beam to be stable, it may not be sufficient. In order to make the oscillation stable, any linear combination of the velocity and displacement of the local electrostatic oscillation has to be bounded. In other words, the eigenvalues of the transfer matrix has to be smaller or equal to 1 as well. The two eigenvalues of the transfer matrix (2.16) can be calculated from the following equations,

$$\lambda_{\pm} = 1 + \xi\left(\cos(\omega_0\tau) - 1\right) \pm \sqrt{\xi \sin(\omega_0\tau)\left[\omega_0\tau(\xi - 1) - \xi\sin(\omega_0\tau)\right]} \tag{2.20}$$

For RHIC parameters, as we have seen above, $\xi << 1$ and equation (2.20) can be rewritten to

$$\lambda_{+} - 1 = \sqrt{-\xi\omega_0\tau\sin(\omega_0\tau)} + O(\xi) \tag{2.21}$$

Therefore, the condition for $\left|\lambda_{+} - 1\right| \le 1$ is $\sin(\omega_0\tau) \ge 0$ or

$$\omega_0\tau \le \pi \tag{2.22}$$

which correspond to the following electron beam bunch length

$$l_e = \frac{N_e}{\pi\sigma_e^2 n_{e,th}\gamma} = 4.589\times10^{-2}\,m$$

Outside the cooling section, the ion beam Plasma oscillation will be described by the following equation,

$$\frac{d^2 s_i}{dt^2} + \omega_{pi}^2 s_i = 0 \tag{2.23}$$

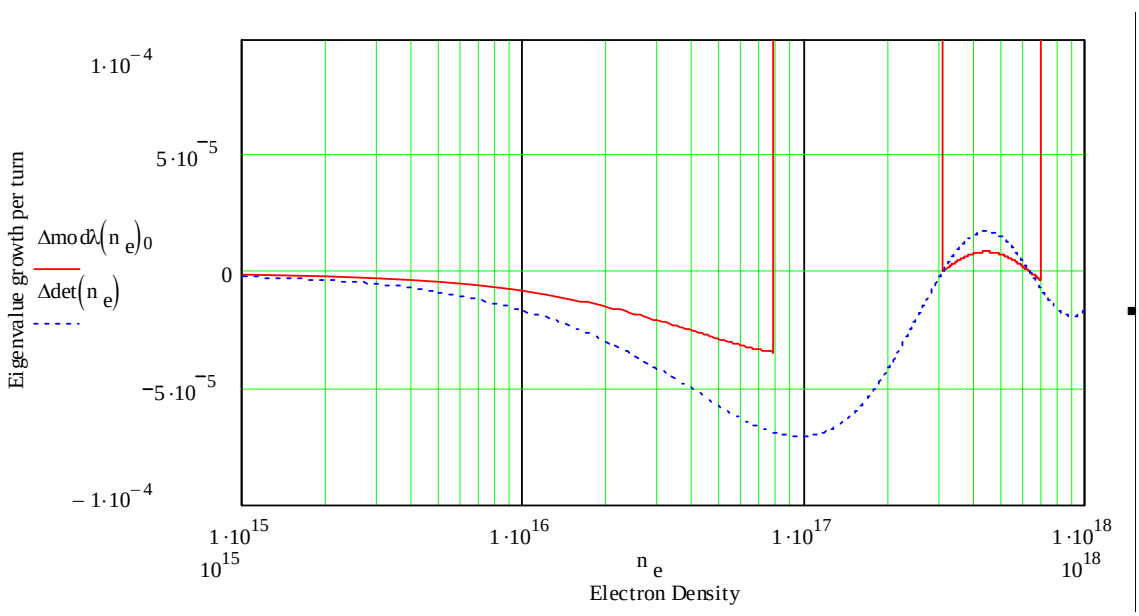


Fig.4 Plots of the eigenvalues and the determinant of the transfer matrix. The x axis is the electron density in units of m^{-3}. The red solid curve is $|\lambda_+|-1$ and the blue dot curve is $|M_{langmuir}|-1$ as already shown in the Fig.3. The maximal value of $|\lambda_+|-1$ is around 0.01, which is much larger than the maximum of $|M_{langmuir}|$, and the threshold happens at $\omega_0\tau=\pi$, i.e. $n_e=7.75\times10^{16}$.

The corresponding transfer matrix is

$$M_{rest}=\begin{pmatrix}\cos(\omega_{pi}\tau_{rest}) & \dfrac{\sin(\omega_{pi}\tau_{rest})}{\omega_{pi}}\\ -\omega_{pi}\sin(\omega_{pi}\tau_{rest}) & \cos(\omega_{pi}\tau_{rest})\end{pmatrix} \tag{2.24}$$

where

$$\tau_{rest}=\frac{C_{RHIC}-l_{cool}}{\gamma c}=1.259\times10^{-7} \tag{2.25}$$

is the flight time outside the electron cooling section and $C_{RHIC}=3833.845m$ is the circumference of RHIC. Thus the one turn transfer matrix for the longitudinal plasma oscillation is

$$M_{ring}=M_{Langmuir}M_{rest} \tag{2.26}$$

As shown in Fig.4.1, including the rest of the ring does not affect the determinant of the one turn transfer matrix but the maximal eigenvalue does change. As a result, the eigenvalue and the determinant set the same limitation to the electron beam density, which for the current magnetized electron cooler design is

$$l_e\geq\frac{N_e}{\pi\sigma_e^2 n_{e,th}\gamma}=1.147\times10^{-2}m$$

The synchrotron tune of RHIC is 3.7×10^{-4} which is 5 times faster than the maximal growth rate, 6.6×10^{-5} per turn. So the oscillation could be distorted by the synchrotron motion before it is actually built up.

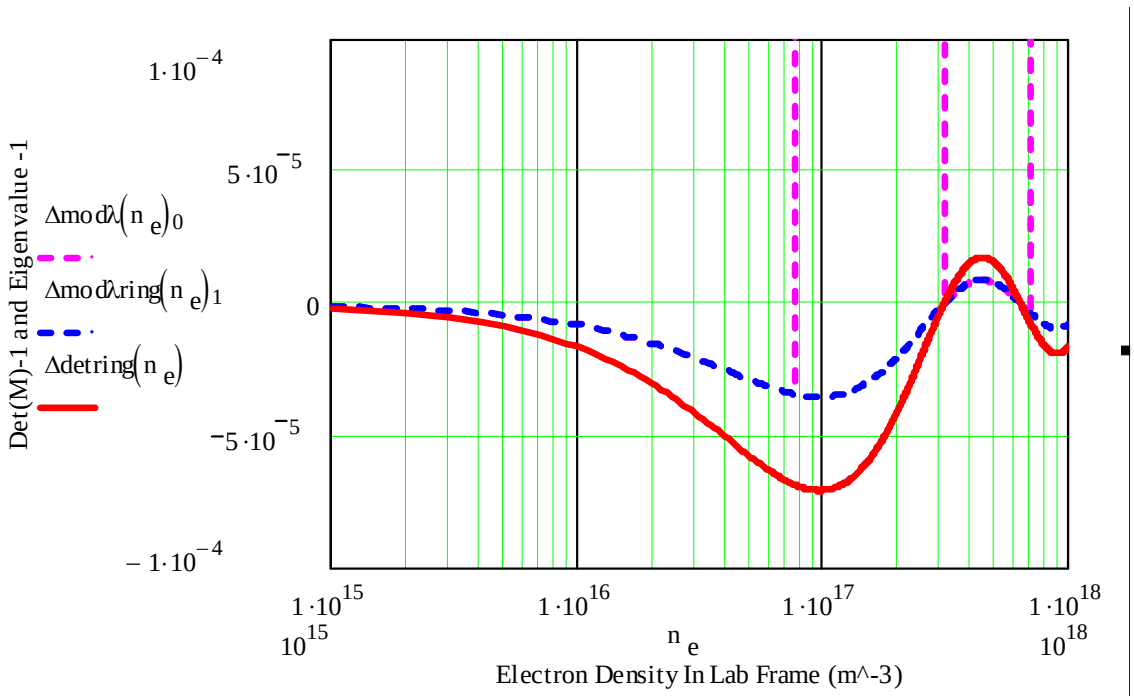


Fig.4.1 Comparison of the determinant and eigenvalues of M_{ring} and $M_{Langmuir}$. The red solid line and the red dot line is the determinant and eigenvalue of $M_{Langmuir}$. The purple dash and blue dash-dot lines are the determinant and eigenvalue of M_{ring}. The x axis is the beam frame electron density in units of m^{-3}.

Section 3. Transverse-Transverse Coupling

3.1 Transversal coupling for nonmagnetized cooling

When the beam enters the cooling section and merges with the cooling electron beam, a misalignment perturbation of the two beams can cause their centroids to perform transversal oscillation as shown in Fig.5. In order to obtain the equation of motion for the beam centroids, let's consider the electrostatic field within the beams in the commoving frame. As mentioned in section 2, in commoving frame, the beams have the geometry as following,

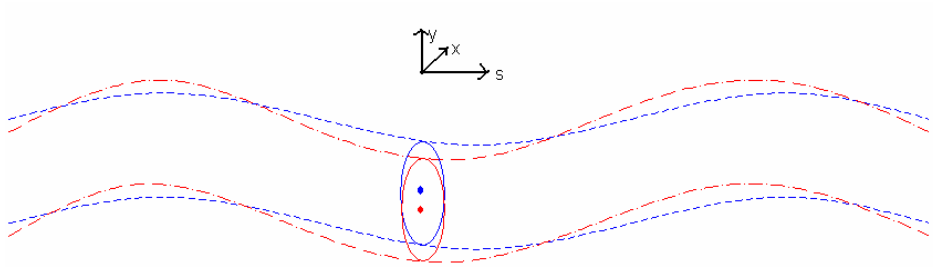
$$l_e' = l_e\gamma = 0.18\times 100 = 18 meter >> \sigma_e = 0.002 meter \qquad (3.1)$$

$$l_i' = l_i\gamma = 0.3\times 100 = 30 meter >> \sigma_i = 0.0014 meter \qquad (3.2)$$

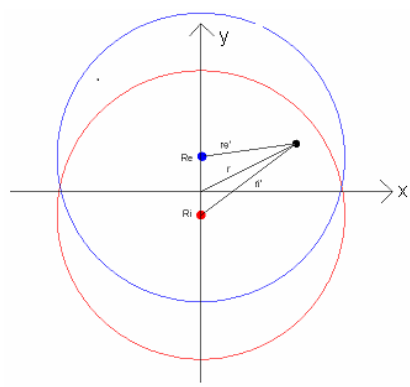
As shown in Fig.5 (b), the coordinates relations among the beam centroids frame and the commoving frame is

$$\vec{r} = \vec{R}_e + \vec{r}_e' = \vec{R}_i + \vec{r}_i' \tag{3.3}$$

where,



(a)



(b)

Fig.5 Illustration of Transversal Coupling in the cooling section. (a) The red dash curve represents for the ion beam and the blue dash-dot curve represents for the electron beam. The two circles represent the cross-section of the two beam and the solid spots are their centroids. (b) The cross sections of the beams shows the coordinates relations , where similar with (a), the solid spots are the beam centroids and $\vec{R}_e, \vec{R}_i$ are their coordinates.

$$\vec{R}_e = \int \vec{r}_e f_e(x,y,z,t)dxdy \qquad \text{(integrate over electron beam cross section)} \tag{3.4}$$

$$\vec{R}_i = \int \vec{r}_i f_e(x,y,z,t)dxdy \qquad \text{(integrate over ion beam cross section)} \tag{3.5}$$

Equation (3.1) and (3.2) show that infinite long beam approximation could be used to calculate the transverse electric field. If the oscillation amplitude is smaller than the beam size, the electric field within the overlapping part of the two beams is

$$\vec{E}_\perp(\vec{r},t) = \frac{-en_e\vec{r}_e'}{2\varepsilon_0} + \frac{Z_i e n_i \vec{r}_i'}{2\varepsilon_0} \tag{3.6}$$

Thus in the rest frame, for an ion/electron sitting in position $\vec{r}$ and in time t, the transversal electrostatic force it sees is

$$\vec{F}_{\perp}^{e}(\vec{r},t)=\frac{e^2 n_e \vec{r}_e'}{2\varepsilon_0}-\frac{Z_i e^2 n_i \vec{r}_i'}{2\varepsilon_0} \qquad \text{(for an electron in } (\vec{r},t)\text{)} \qquad (3.7)$$

$$\vec{F}_{\perp}^{i}(\vec{r},t)=-\frac{Z_i e^2 n_e \vec{r}_e'}{2\varepsilon_0}+\frac{Z_i^2 e^2 n_i \vec{r}_i'}{2\varepsilon_0} \qquad \text{(for an ion in } (\vec{r},t)\text{)} \qquad (3.8)$$

And its equation of motion is

$$\frac{d^2}{dt^2}\vec{r}_e'=\omega_{pe}^2\vec{r}_e'-\omega_{ei}^2(\vec{r}_e-\vec{R}_i) \qquad \text{(for an electron in } (\vec{r}_e,t)\text{)} \qquad (3.9)$$

$$\frac{d^2}{dt^2}\vec{r}_i=-\omega_{ie}^2(\vec{r}_i-\vec{R}_e)+\omega_{pi}^2\vec{r}_i' \qquad \text{(for an ion in } (\vec{r}_i,t)\text{)} \qquad (3.10)$$

Assuming the beam distribution function changes slowly with time and by integrating equation (3.9) and (3.10) over the cross section according to the beam distribution, the centroids' equations of motion can be obtained as following,

$$\frac{d^2}{dt^2}\vec{R}_e(z,t)+\omega_{ei}^2\vec{R}_e(z,t)=\omega_{ei}^2\vec{R}_i(z,t) \qquad (3.11)$$

$$\frac{d^2}{dt^2}\vec{R}_i(z,t)+\omega_{ie}^2\vec{R}_i(z,t)=\omega_{ie}^2\vec{R}_e(z,t) \qquad (3.12)$$

The transversal commoving frame plasma frequencies, $\omega_{pi},\omega_{pe},\omega_{ie},\omega_{ei}$ are defined as .

$$\omega_{pe}=\sqrt{\frac{n_e e^2}{2m_e\varepsilon_0}} \qquad (3.12.3)$$

$$\omega_{pi}=\sqrt{\frac{Z_i^2 n_i e^2}{2M_i\varepsilon_0}} \qquad (3.12.4)$$

$$\omega_{ie}=\sqrt{\frac{Z_i n_e e^2}{2M_i\varepsilon_0}} \qquad (3.12.5)$$

$$\omega_{ei}=\sqrt{\frac{Z_i n_i e^2}{2m_e\varepsilon_0}}=3.108\times10^8\,s^{-1} \qquad (3.12.6)$$

Comparing with equation (2.4) and (2.5), the only difference is the coefficients of the second terms at the left hand side both for ion and electron beam. So the steps for solving (3.11) and (3.12) are similar with what has been done in section 2.2. By setting the initial condition,

$$\vec{R}_e(z,0)=\frac{d}{dt}\vec{R}_e(z,0)=0$$

one gets

$$\frac{d^2}{dt^2}\vec{R}_i(z,t) = -\omega_{ie}^2\left[\vec{R}_i(z,0)\cos(\omega_0 t) + \frac{1}{\omega_0}\dot{\vec{R}}_i(z,0)\sin(\omega_0 t)\right] \tag{3.17}$$

where ω_0 is now defined as

$$\omega_0 = \sqrt{\omega_{ie}^2 + \omega_{ei}^2} \tag{3.17.1}$$

Integrating equation (3.17) from the front side of the cooling section $t = 0$, one obtains

$$\frac{d}{dt}\vec{R}_i(z,t) = -\xi\omega_0 \sin(\omega_0 t)\vec{R}_i(z,0) + \left[1 + \xi(\cos(\omega_0 t) - 1)\right]\dot{\vec{R}}_i(z,0) \tag{3.18}$$

$$\vec{R}_i(z,\tau) = \left[1 + \xi(\cos(\omega_0 \tau) - 1)\right]\vec{R}_i(z,0) + \frac{1}{\omega_0}\left[\omega_0\tau(1-\xi) + \xi\sin(\omega_0\tau)\right]\dot{\vec{R}}_i(z,0) \tag{3.19}$$

where ξ is now defined as

$$\xi = \frac{\omega_{ie}^2}{\omega_0^2} \tag{3.20}$$

Thus the transfer matrix of two stream dipole type transversal interaction for the ion beam centroid is

$$M_{transverse} = \begin{pmatrix} \xi(\cos(\omega_0\tau) - 1) + 1 & \frac{1}{\omega_0}\left[\xi\sin(\omega_0\tau) + (1-\xi)\omega_0\tau\right] \\ -\xi\omega_0\sin(\omega_0\tau) & \xi(\cos(\omega_0\tau) - 1) + 1 \end{pmatrix} \tag{3.22}$$

which has exactly the same form of the transfer matrix due to the longitudinal Langmuir oscillation as shown in equation (2.16) except that the ξ and ω_0 are defined differently from section 2.2. Thus for each turn passing through the cooling section, the transversal centroid motion is effected by the electron beam according to the following expression,

$$\begin{pmatrix} R_i \\ \dot{R}_i \end{pmatrix}_\tau = M_{transverse}\begin{pmatrix} R_i \\ \dot{R}_i \end{pmatrix}_0 \tag{3.23}$$

The calculation of $|M_{transverse}|$ is the same as in equation (2.17) and (2.18)

$$|M_{transverse}| - 1 = \frac{4\omega_{ie}^2\omega_{ei}^2}{\omega_0^4}\sin^2\left(\frac{\omega_0\tau}{2}\right)\left[\frac{\omega_0\tau}{2}\cot\left(\frac{\omega_0\tau}{2}\right) - 1\right] \tag{3.24}$$

As shown in Fig.6[1], since the oscillation frequency for the transverse oscillation , ω_0, is 5 orders smaller than the longitudinal Langmuir oscillation, the instability threshold is 5 orders larger than what the longitudinal oscillation has and thus is not likely to be a real

[1] All the Figures in this subsection are given for the commoving frame densities $n_i = 5.3\times10^{12}\,m^{-3}$ and $n_e = 3.3\times10^{15}\,m^{-3}$. The more realistic calculation for non-magnetized electron cooling design with wiggler field will be given in subsetion 3.4.

limitation for the electron cooler design. The eigenvalues of $M_{transverse}$ is also the same as (2.20) with different definitions of ξ and ω_0.

$$\lambda_{\pm} = 1 + \xi\left(\cos(\omega_0\tau) - 1\right) \pm \sqrt{\xi \sin(\omega_0\tau)\left[\omega_0\tau(\xi - 1) - \xi \sin(\omega_0\tau)\right]} \tag{3.25}$$

Comparing with the longitudinal oscillation, the instability threshold of the electron beam density is pretty much the same for the eigenvalue restriction and the determinant restriction as shown in Fig.7. To implement $M_{transverse}$ into the ring, one need to do the Lorentz transformation at the entrance and inverse Lorentz transformation at the end of the cooling section since $M_{transverse}$ is derived in the commoving frame. Furthermore, to avoid double counting the phase advance inside the cooling section, one may add negative drift matrix to compensate. As a result, the transfer matrix in lab frame is given by

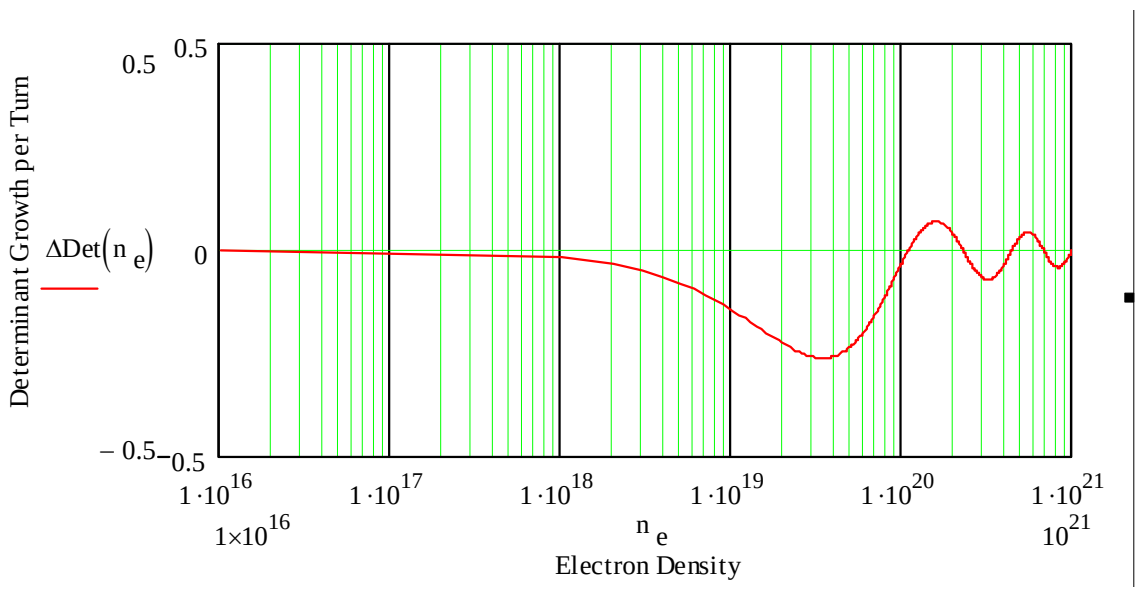


Fig. 6 The dependence of $\left|M_{transverse}\right| - 1$ on the electron density n_e shows a instability threshold 5 orders larger than the longitudinal Langmuir oscillation.

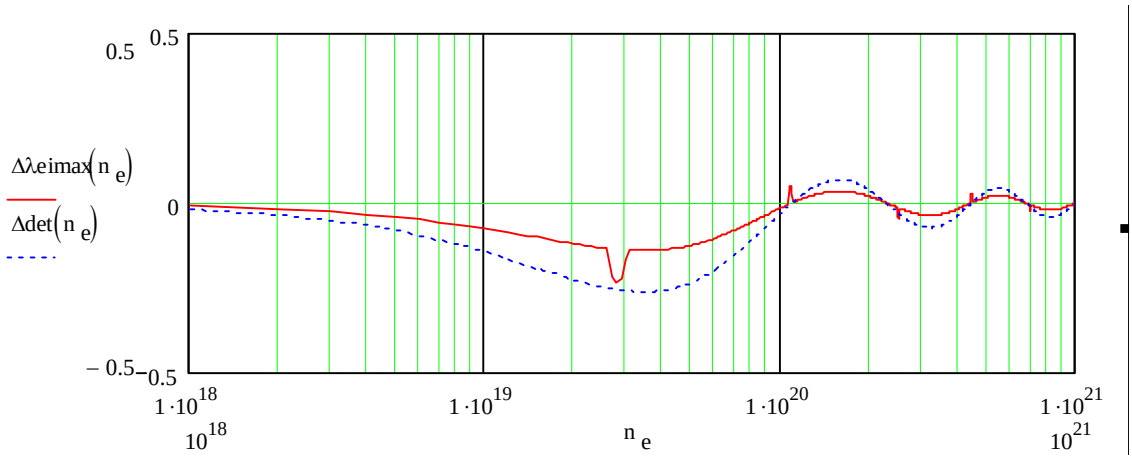


Fig. 7 Plot of $|\lambda_+|-1$ and $|M_{transverse}|-1$ for the transverse dipole type oscillation. The x axis is the electron density in units of m^{-3}. The red solid curve is $|\lambda_+|-1$ and the blue dot curve is $|M_{transverse}|-1$.

$$M_{tranlab} = R_{Twiss} L_{drift}(-\frac{l_{cool}}{2}) L_{lorentz}{}^{-1} M_{transverse} L_{lorentz} L_{drift}(-\frac{l_{cool}}{2})$$

where

$$R_{Twiss} = \begin{pmatrix} \cos(2\pi\nu_{x,y}) + \alpha_{x,y} \cdot \sin(2\pi\nu_{x,y}) & \beta_{x,y}\sin(2\pi\nu_{x,y}) \\ \dfrac{-(1+\alpha_{x,y}{}^2)\sin(2\pi\nu_{x,y})}{\beta_{x,y}} & \cos(2\pi\nu_{x,y}) - \alpha_{x,y} \cdot \sin(2\pi\nu_{x,y}) \end{pmatrix}$$

$$L_{lorentz} = \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}$$

$$L_{drift}(-\frac{l_{cool}}{2}) = \begin{pmatrix} 1 & -\frac{l_{cool}}{2} \\ 0 & 1 \end{pmatrix}$$

$$\alpha_{x,y} = -\frac{\beta_{x,y}'}{2}$$

As shown in Fig.8, the threshold of the instability decreases about one order of magnitude after including the cooling section into the ring.

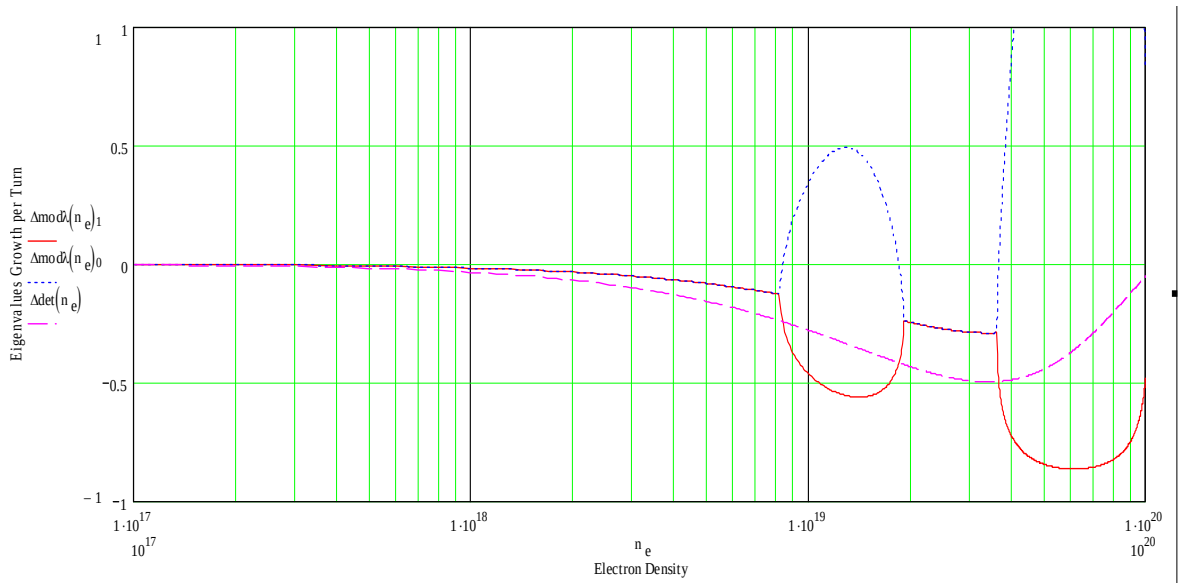
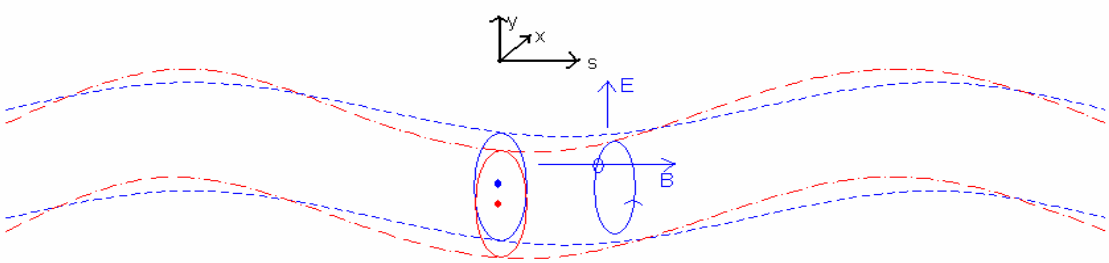


Fig. 8 Plot of $|\lambda_{tranlab}|-1$ and $|M_{tranlab}|-1$ for the transverse dipole type oscillation. The x axis is the electron density in units of m^{-3} (in beam frame). The red solid curve is $|M_{tranlab}|-1$. The blue dot curve and the purple solid curve are the two eigenvalues of $M_{tranlab}$,i.e. $|\lambda_{tranlab}|-1$. The tune has been taken as $\nu_x = 28.23$ and the betatron function is taken as $\beta_x = 40 meter$

3.2 Transversal Coupling in the presence of a solenoid

For magnetized cooling, a solenoid with strong longitudinal magnetic field has to be included in the cooling section. For RHIC electron cooler, one option is to include a 30 meter long $B_{//} = 5T$ solenoid to enhance the cooling force. The Larmor frequencies in the beam commoving frame for the ions and electrons are



Transverse dipole oscillation in the cooling section with longitudinal magnetic field. The small blue circle crossed by the magnetic field line represents the electron Larmor oscillation orbit and the bigger circle going through it with an arrow represents the drift motion orbit.

$$\omega_{ce} = \frac{eB_{//}}{m_e} = 8.79 \times 10^{11} s^{-1} \tag{3.26}$$

$$\omega_{ci} = \frac{Z_i eB_{//}}{M_i} = 1.93 \times 10^{8} s^{-1} \tag{3.27}$$

Consequently, only the electrons are completely magnetized as the flight time is $10^{-9} s$. The equation of motion for each ion or electron is similar to (3.9) or (3.10) with an additional term coming from the magnetic force, i.e.

$$\frac{d^2}{dt^2}\vec{r}_e = \omega_{pe}^2(\vec{r}_e - \vec{R}_e) - \omega_{ei}^2(\vec{r}_e - \vec{R}_i) - \omega_{ce}(\frac{d}{dt}\vec{r}_e \times \hat{s}) \quad (3.28)$$

$$\frac{d^2}{dt^2}\vec{r}_i = -\omega_{ie}^2(\vec{r}_i - \vec{R}_e) + \omega_{pi}^2(\vec{r}_i - \vec{R}_i) + \omega_{ci}(\frac{d}{dt}\vec{r}_i \times \hat{s}) \quad (3.29)$$

where $\hat{s}$ is the unit vector along the longitudinal direction. Integrating (3.28) and (3.29) over the electron and ion beam transverse distribution respectively, one gets

$$\frac{d^2}{dt^2}\vec{R}_e + \omega_{ei}^2\vec{R}_e + \omega_{ce}(\frac{d}{dt}\vec{R}_e \times \hat{s}) = \omega_{ei}^2\vec{R}_i \quad (3.30)$$

$$\frac{d^2}{dt^2}\vec{R}_i + \omega_{ie}^2\vec{R}_i - \omega_{ce}(\frac{d}{dt}\vec{R}_i \times \hat{s}) = \omega_{ie}^2\vec{R}_e \quad (3.31)$$

Define,

$$Z_e \equiv X_e + iY_e \quad (3.38)$$

$$Z_i \equiv X_i + iY_i \quad (3.39)$$

where X and Y are the transversal components of R.

Thus equation (3.30) and (3.31) can be rewritten as

$$\frac{d}{dt}\overline{Z_e} + i\Lambda\overline{Z_e} = i\Lambda Z_i \quad (3.50)$$

$$\frac{d^2}{dt^2}Z_i + \omega_{ie}^2 Z_i + i\omega_{ci}\frac{d}{dt}Z_i = \omega_{ie}^2\overline{Z_e} \quad (3.52)$$

where

$$\Lambda = \frac{\omega_{ei}^2}{\omega_{ce}} = 7.57 \times 10^5 s^{-1} \quad (3.51)$$

$$\overline{Z_e} = \frac{1}{T_{ce}}\int_0^{T_{ce}} Z_e(t)dt$$

Equation (3.50) and (3.52) describe the coupling of the ion beam centroid with the guiding center of the electron beam centroid. Taking the trial solutions as the following

$$Z_i(t) = a_i e^{-i\omega t} \quad (3.53)$$

$$\overline{Z_e}(t) = a_e e^{-i\omega t} \quad (3.54)$$

and inserting them into (3.50) and (3.52) respectively, one gets

$$(-i\omega + i\Lambda)a_e - i\Lambda a_i = 0 \quad (3.55)$$

$$\left(-\omega^2+\omega_{ie}^2+\omega_{ci}\omega\right)a_i-\omega_{ie}^2 a_e=0 \tag{3.56}$$

Thus the eigenfrequencies are[1]

$$\omega_{1,2}=\frac{1}{2}\left[\left(\omega_{ci}+\Lambda\right)\pm\sqrt{\left(\omega_{ci}+\Lambda\right)^2+4\left(\omega_{ie}^2-\Lambda\omega_{ci}\right)}\right] \tag{3.57.1}$$

$$\omega_3=0 \tag{3.57.3}$$

Thus, the solution of (3.50) and (3.52) should be the linear combination of three modes with the eigenfrequencies ω_0, ω_1 and ω_2 respectively, i.e.

$$Z_i(t)=\sum_{\alpha=1}^{3}a_{i\alpha}e^{-i\omega_\alpha t} \tag{3.61}$$

$$\overline{Z_e}(t)=\sum_{\alpha=1}^{3}a_{e\alpha}e^{-i\omega_\alpha t} \tag{3.62}$$

From equation (3.56), one gets

$$a_{e\alpha}=\left(1-\frac{\omega_\alpha^2}{\omega_{ie}^2}+\frac{\omega_{ci}\omega_\alpha}{\omega_{ie}^2}\right)a_{i\alpha}=T_\alpha a_{i\alpha} \tag{3.63}$$

where

$$T_\alpha=\left(1-\frac{\omega_\alpha^2}{\omega_{ie}^2}+\frac{\omega_{ci}\omega_\alpha}{\omega_{ie}^2}\right) \tag{3.64}$$

By using equation (3.63), equation (3.62) can rewritten as

$$\overline{Z_e}(t)=\sum_{\alpha=1}^{3}T_\alpha a_{i\alpha}e^{-i\omega_\alpha t} \tag{3.65}$$

Taking the derivative of (3.61) with respect to t, one gets

$$\frac{d}{dt}Z_i(t)=\sum_{\alpha=1}^{3}(-i\omega_\alpha)a_{i\alpha}e^{-i\omega_\alpha t} \tag{3.66}$$

When $t=0$, equation (3.61), (3.65) and (3.66) can be used to determine the coefficient $a_{i\alpha}$ and the solution for equation (3.50) and (3.52) can be obtained as,

$$\begin{pmatrix} Z_i(t) \\ \dot{Z}_i(t) \\ \overline{Z_e}(t) \end{pmatrix}=M\begin{pmatrix} Z_i(0) \\ \dot{Z}_i(0) \\ \overline{Z_e}(0) \end{pmatrix} \tag{3.71}$$

where

---

[1] For $B_{//}=5T$, $n_i=7.697\times10^{11}m^{-3}$ and $n_e=7.117\times10^{14}m^{-3}$, $\omega_+=1.946\times10^8 s^{-1}$ $\omega_-=-1.169\times10^6 s^{-1}$

$$M = \begin{pmatrix} e^{-i\omega_1 t} & e^{-i\omega_2 t} & 1 \\ -i\omega_1 e^{-i\omega_1 t} & -i\omega_2 e^{-i\omega_2 t} & 0 \\ T_1 e^{-i\omega_1 t} & T_2 e^{-i\omega_2 t} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ -i\omega_1 & -i\omega_2 & 0 \\ T_1 & T_2 & 1 \end{pmatrix}^{-1} \qquad (3.72)$$

Equation (3.57.3) has been taken into account to get a simpler form of M in (3.72). Setting the initial condition of the electron to be $\overline{Z_e}(0) = 0$, the solution for the ion beam centroid can be expressed as a 2×2 transfer matrix, i.e.

$$\begin{pmatrix} Z_i(t) \\ \dot{Z}_i(t) \end{pmatrix} = M_{ion} \begin{pmatrix} Z_i(0) \\ \dot{Z}_i(0) \end{pmatrix} \qquad (3.74)$$

where

$$M_{ion} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \qquad (3.75)$$

The matrix elements of M_{ion} are listed in Appendix2, Equation (3.75) can be rewritten into a 4×4 matrix form in Frenet-Serret coordinate system as,

$$\begin{pmatrix} X_i(t) \\ X_i'(t) \\ Y_i(t) \\ Y_i'(t) \end{pmatrix} = T_{cool}^{'} \begin{pmatrix} X_i(0) \\ X_i'(0) \\ Y_i(0) \\ Y_i'(0) \end{pmatrix} \qquad (3.90)$$

where

$$T_{cool}^{'} = \begin{pmatrix} A_{11} & A_{12}V_{//} & -B_{11} & -B_{12}V_{//} \\ \frac{A_{21}}{V_{//}} & A_{22} & \frac{-B_{21}}{V_{//}} & -B_{22} \\ B_{11} & B_{12}V_{//} & A_{11} & A_{12}V_{//} \\ \frac{B_{21}}{V_{//}} & B_{22} & \frac{A_{21}}{V_{//}} & A_{22} \end{pmatrix} \qquad (3.91)$$

The matrix elements $A_{i,j}$ and $B_{i,j}$ are listed in the Appendix 2[1]. In order to obtain the one turn betatron oscillation transfer matrix, consider the ion beam transverse motion starting from the front end of the solenoid. As the beam going through the front end, it is affected by the fringe field and the effects can be represented by the transfer matrix[2]

---

[1] For $B_{//} = 5T$, $\gamma = 100$, $n_i = 7.697 \times 10^{11} m^{-3}$, $n_e = 7.117 \times 10^{14} m^{-3}$ and 60 meter long cooling section, the transfer matrix $T_{cool}^{'}$ can be calculated as shown below,

$$T_{cool}^{'} = \begin{pmatrix} 1 & 0.585 & -6.376\times10^{-5} & 0.115 \\ -1.618\times10^{-3} & 0.926 & -3.172\times10^{-4} & 0.377 \\ 6.376\times10^{-5} & -0.115 & 1 & 0.585 \\ 3.172\times10^{-4} & -0.377 & -1.618\times10^{-3} & 0.926 \end{pmatrix}$$

[2] Reader should not confused the charge number Z_i in the following expression with the complex coordinates defined in (3.39)

$$E_C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{iZ_i eB_{//}}{2M_i \gamma V_{//}} & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{iZ_i eB_{//}}{2M_i \gamma V_{//}} & 0 & 0 & 1 \end{pmatrix}$$

Then the beam need to be transferred into commoving frame since $T_{cool}^{'}$ is derived in the commoving frame. The Lorenz transfer matrix for the transverse plane is given by

$$L_{lorentz} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}$$

Inside the solenoid, the ion beam sees the electron beams and the longitudinal magnetic field, whose effects to the ion beam centroid have been described by the transfer matrix $T_{cool}^{'}$ defined in (3.91). At the end of the solenoid, the beam has to be transformed back to the lab frame since the edge field effects and the Twiss matrix are all given in the lab frame. As the ion beam getting out the solenoid, it sees the fringe field again but in the opposite direction, whose effects is described by E_C^{-1}. Since the drift effects inside the solenoid has been considered in $T_{cool}^{'}$ already, the Twiss matrix should not include it again. However, when the beam optics code calculates the betatron tune of the accelerator with a cooling section, it automatically takes the cooling section drift into account and thus it is necessary to exclude the cooling section drift from the Twiss matrix. This exclusion can be done by inserting the drift transfer matrix for negative half solenoid length

$$L_{drift}\left(-\frac{L}{2}\right) = \begin{pmatrix} 1 & -\frac{L}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{L}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

on both sides of the Twiss matrix and thus keeping the symmetry of the accelerator. L is the length of the solenoid and for RHIC electron cooler,

$$L = 60 meter$$

The Twiss matrix for the whole ring without considering the electron-ion beam coupling is given by

$$R_{Twiss} = \begin{pmatrix} \cos(2\pi\nu_x) & \beta_x \sin(2\pi\nu_x) & 0 & 0 \\ -\dfrac{\sin(2\pi\nu_x)}{\beta_x} & \cos(2\pi\nu_x) & 0 & 0 \\ 0 & 0 & \cos(2\pi\nu_y) & \beta_y \sin(2\pi\nu_y) \\ 0 & 0 & -\dfrac{\sin(2\pi\nu_y)}{\beta_y} & \cos(2\pi\nu_y) \end{pmatrix}$$

where ν_x, ν_y are the betatron tune including the cooling section drift and β_x, β_y are the horizontal and vertical betatron functions at the back end of the cooling section. Thus, the one turn betatron oscillation transfer matrix is given by

$$T_{ring} = L_{drift} R_{Twiss} L_{drift} E_C^{-1} L_{lorentz}^{-1} T'_{cool} L_{lorentz} E_C \tag{3.92}$$

For RHIC,

$$\begin{aligned} \nu_x &= 28.19 \\ \nu_y &= 29.18 \\ \beta_x &= \beta_y = 60 meter \end{aligned} \tag{3.93}$$

The determinant of T_{ring} for the current RHIC parameters is calculated to be [1]

$$\left|T_{ring}\right| - 1 = 7.02 \times 10^{-9} \tag{3.95}$$

And the eigenvalues of T_{ring} are

$$\lambda = \begin{pmatrix} 0.191 + 0.982i \\ 0.191 - 0.982i \\ 0.551 + 0.834i \\ 0.551 - 0.834i \end{pmatrix} \tag{3.96}$$

The amplitudes of the eigenvalues is always slightly different from one and for (3.96), they are

$$|\lambda| - 1 = \sqrt{\mathrm{Re}(\lambda)^2 + \mathrm{Im}(\lambda)^2} - 1 = \begin{pmatrix} -5.106 \times 10^{-8} \\ -5.106 \times 10^{-8} \\ 5.457 \times 10^{-8} \\ 5.457 \times 10^{-8} \end{pmatrix} \tag{3.97}$$

---

[1] For $B_{//} = 5T$, $\gamma = 100$, $n_i = 7.697 \times 10^{11} m^{-3}$, $n_e = 7.117 \times 10^{14} m^{-3}$ and 60 meter long solenoid, the transfer matrix T_{ring} can be calculated as shown below,

$$T_{ring} = \begin{pmatrix} 0.789 & 67.748 & 0.155 & 13.271 \\ -1.544 \times 10^{-2} & -0.105 & -3.024 \times 10^{-3} & -2.052 \times 10^{-2} \\ -0.164 & -12.925 & 0.836 & 65.981 \\ 2.951 \times 10^{-3} & 7.24 \times 10^{-3} & -1.506 \times 10^{-2} & -3.696 \times 10^{-2} \end{pmatrix}$$

The maximum eigenvalue amplitude is very close to the approximate analytical formula given by V.Parkhomchuk for short interaction time,

$$\Delta\lambda_{\max} = \sqrt{1+\frac{1}{2}\frac{\beta_i\omega_{ie}^2\Lambda\tau^2}{\gamma V_{//}}} - 1 = 5.476\times10^{-8} \tag{3.98}$$

where Λ is given in (3.51) and $\tau = \frac{l}{\gamma c}$ is the flight time in commoving frame. As shown in Fig.9 and Fig 10, for the considered ion beam and lattice parameters, the determinant of the transfer matrix and the maximum eigenvalue amplitude are always bigger than 1, which can cause the betatron oscillation amplitude increase from turn to turn. For the current parameters, the growth rate is

$$\begin{aligned}\Gamma &= \frac{(|\lambda|_{\max} - 1)}{T_{rev}} \\ &= 4.3\times10^{-3}\,s^{-1}\end{aligned} \tag{3.99}$$

where T_{rev} is the revolution frequency and $C_{rhic} = 3833.845 meter$ is the circumstance of RHIC. The growth time is thus

$$t_{rise} = \frac{1}{\Gamma} = 233\sec ond \tag{3.100}$$

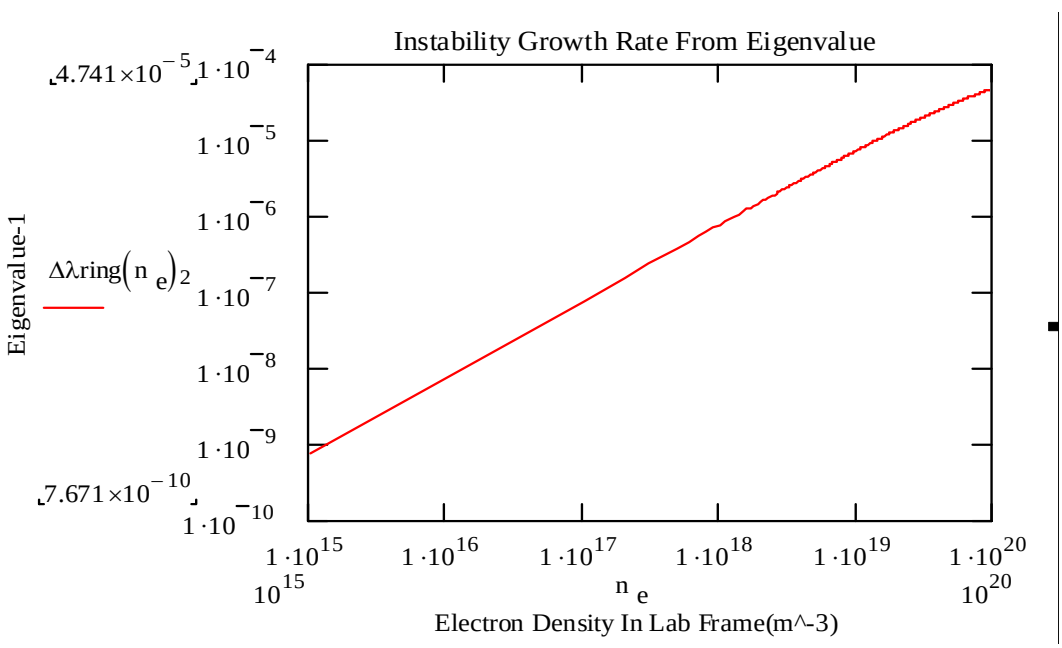


Fig.9 The dependence of the eigenvalue amplitude on the electron density. The x axis is the electron density in commoving frame, and the units is $meter^{-3}$ the y axis is the maximum value of $|\lambda|-1$ as defined in equation (3.97)

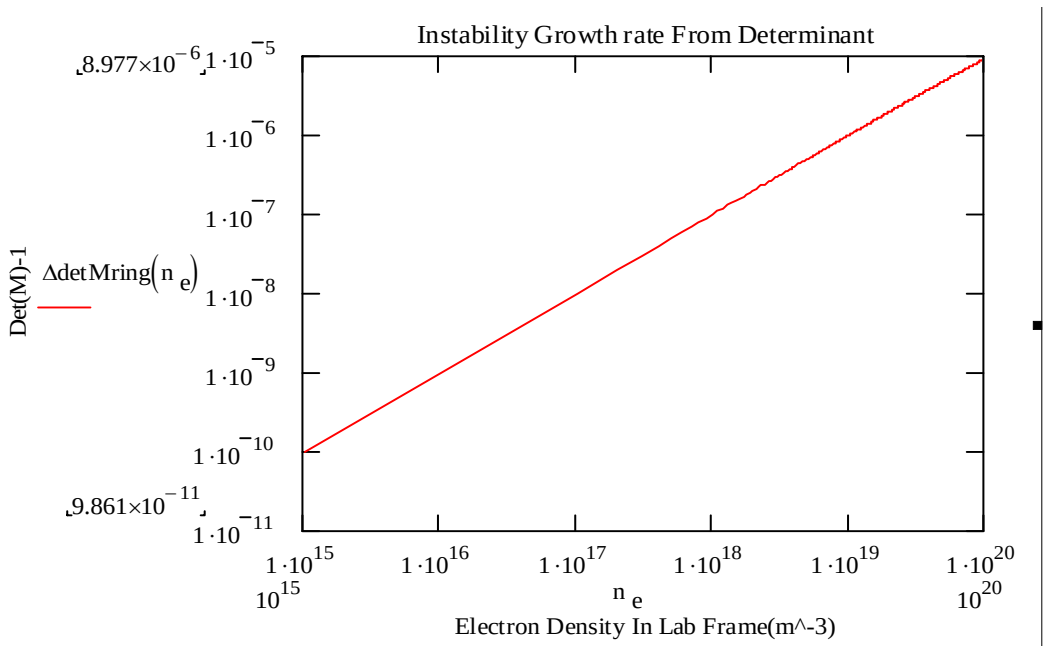


Fig.10 The dependence of the determinant on the electron density. The x axis is the electron density in commoving frame, and the units is $meter^{-3}$. The y axis is $|T_{ring}|-1$ as defined in equation (3.95).

Many facilities such as NAP-M, Fermi lab, Indiana, TARN II and COSY has observed the transverse coherent instability induced by the electron-ion coherent interaction and different methods have been applied against it. For the dipole instabilities, a feed back system is efficient to damp the transverse oscillation amplitude. In the Fermi lab recycler, the instabilities stops after the machine have been decoupled for horizontal and vertical motions within the cooling section. For RHIC electron cooler, since there is no solenoid in the cooling section, this instability will not take place (Ref. Section 3.4).

Section 3.3 Ion clouds effects for the transverse coherent instability within a solenoid

If the negative charge from the electron beam is bigger than the positive charge from the circulating ion beam[1], the ions produced from the residue gas can accumulate inside the cooling section unless the incoming beams make their motion unstable. Driven by the electron and ion beams, the accumulating ion clouds could oscillate and act back to the circulating beams. In section 3.3.1, the ion clouds motion inside a solenoid has been

[1] For magnetized electron cooling scheme, the electron charge per bunch is 20nC, which is indeed bigger than the gold ion beam charge, 13nC. However, for the non-magnetized electron cooling scheme, the electron charge per bunch is 5 nC and the ion clouds can't accumulate within the cooling section.

studied and stability condition has been shown for varies magnetic field strength. In section 3.3.2, the effects of ion clouds to the transverse coherent oscillation have been analyzed.

3.3.1Ion clouds motion in the cooling section

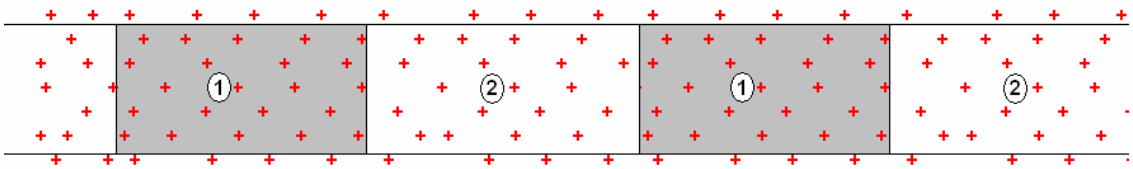


Fig.11 Illustration of the ion clouds inside the solenoid. The red '+' represents the ion cloud and the filled gray region marked '1' represents the incoming commoving electron and ion bunches and blank region '2' represents the space between two successive bunches.

For the first order approximation, assume the displacement of the beam centroid is small compared with the beam size and can be ignored for the moment. For simplicity, we also assume the electron bunch has the same bunch length with the ion bunch (This assumption will not make the result different from the real case since the ion motion will only depend on the total electron charge per bunch). The equation of motion for a single accumulated ion in region 1 (where the beams are present) is

$$\frac{d^2}{dt^2}z_{cl} + i\omega_{ccl}\frac{d}{dt}z_{cl} + (\omega_{cle}^2 - \omega_{cli}^2)z_{cl} = 0 \qquad (3.101)$$

where z_{cl} describes single accumulated ion transverse position and defined as

$$z_{cl} \equiv x_{cl} + iy_{cl} \qquad (3.102)$$

Equation (3.101) is writing in the lab frame since the ion clouds longitudinal motion is slow. The Larmor frequency and the plasma frequencies are defined as[1]

$$\omega_{ccl} = \frac{Z_{cl}eB_{//}}{m_{cl}} = 4.82\times10^8 s^{-1} \qquad (3.103)$$

$$\omega_{cle} = \sqrt{\frac{Z_{cl}n_e e^2}{2\varepsilon_0 m_{cl}}} = 2.4\times10^8 s^{-1} \qquad (3.104)$$

$$\omega_{cli} = \sqrt{\frac{Z_i Z_{cl} n_i e^2}{2\varepsilon_0 m_{cl}}} = 1.91\times10^8 \qquad (3.105)$$

[1] All the numbers given in this section are for $B_{//} = 5T$, $n_e = 6.63\times10^{16} m^{-3}$, $n_i = 5.30\times10^{14} m^{-3}$ and for hydrogen ion, i.e. $Z_{cl} = 1$, $m_{cl} = m_p$ (proton mass).

Setting the trial solution of (3.101) to be

$$z_{cl}(t) = a_{cl} e^{-i\omega t} \tag{3.106}$$

and Inserting (3.106) into (3.101), one get,

$$\omega^2 - \omega_{ccl}\omega - (\omega_{cle}^2 - \omega_{cli}^2) = 0 \tag{3.107}$$

Thus, the eigenfrequencies are

$$\omega_{1,2} = \frac{1}{2}\left[\omega_{ccl} \pm \sqrt{\omega_{ccl}^2 + 4(\omega_{cle}^2 - \omega_{cli}^2)}\right] = \begin{cases} 1.01\times 10^9 s^{-1} \\ -5.33\times 10^8 s^{-1} \end{cases} \tag{3.108}$$

There are two modes for the accumulated ion oscillation with frequency ω_1 and ω_2 respectively. So, (3.106) should be rewritten as the superposition of these two modes.

$$z_{cl}(t) = \sum_{\alpha=1}^{2} a_{cl,\alpha} e^{-i\omega_\alpha t} \tag{3.109}$$

Set the initial condition at $t = 0$ to be

$$z_{cl} = z_{cl}(0) \tag{3.110}$$

$$\dot{z}_{cl} = \dot{z}_{cl}(0) \tag{3.111}$$

From equation (3.109)-(3.111), one gets

$$\begin{pmatrix} z_{cl}(t) \\ \dot{z}_{cl}(t) \end{pmatrix} = M_{focus} \begin{pmatrix} z_{cl}(0) \\ \dot{z}_{cl}(0) \end{pmatrix} \tag{3.116}$$

where M_{focus} is the transfer matrix for the effects of the beams acting on the accumulating ion clouds and is defined as

$$M_{focus} = \frac{1}{\omega_1 - \omega_2} \begin{pmatrix} \omega_1 e^{-i\omega_2 t} - \omega_2 e^{-i\omega_1 t} & -i(e^{-i\omega_2 t} - e^{-i\omega_1 t}) \\ -i\omega_1\omega_2(e^{-i\omega_2 t} - e^{-i\omega_1 t}) & \omega_1 e^{-i\omega_1 t} - \omega_2 e^{-i\omega_2 t} \end{pmatrix} \tag{3.117}$$

In region 2 (the space between two successive bunches), ignoring self field interaction, the accumulated ions only see the longitudinal magnetic field and thus their equation of motion is

$$\frac{d^2}{dt^2} z_{cl} + i\omega_{ccl} \frac{d}{dt} z_{cl} = 0 \tag{3.118}$$

Integrating equation (3.118), one gets

$$\frac{dz_{cl}(t)}{dt} = -i\omega_{ccl}\left[z_{cl}(t) + i\frac{\dot{z}_{cl}(0)}{\omega_{ccl}} - z_{cl}(0) \right] \tag{3.119}$$

and

$$z_{cl}(t) = i\frac{\dot{z}_{cl}(0)}{\omega_{ccl}}e^{-i\omega_{ccl}t} - i\frac{\dot{z}_{cl}(0)}{\omega_{ccl}} + z_{cl}(0) \tag{3.120}$$

Taking the derivative of equation (3.120) with respect to t, the velocity of the accumulated ion is

$$\dot{z}_{cl}(t) = \dot{z}_{cl}(0)e^{-i\omega_{ccl}t} \tag{3.121}$$

From (3.120) and (3.121), the motion of the accumulated ion can be written into the following matrix form,

$$\begin{pmatrix} z_{cl}(t) \\ \dot{z}_{cl}(t) \end{pmatrix} = M_{Larmor}\begin{pmatrix} z_{cl}(0) \\ \dot{z}_{cl}(0) \end{pmatrix} \tag{3.122}$$

where M_{lamour} is the transfer matrix for the Larmor oscillation when the accumulated ion sitting between two bunches and defined as

$$M_{Larmor} = \begin{pmatrix} 1 & i\dfrac{e^{-i\omega_{ccl}t} - 1}{\omega_{ccl}} \\ 0 & e^{-i\omega_{ccl}t} \end{pmatrix} \tag{3.123}$$

From equation (3.116) and (3.122), the transfer matrix for one whole bunch period (the time interval for two successive bunches passing by) is

$$\begin{aligned} M_{fl} &= M_{focus}(t_1)M_{Larmor}(t_2) \\ &= \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \end{aligned} \tag{3.124}$$

where t_1 and t_2 are the bunch length and the spacing between bunches respectively. The elements of M_{fl} are defined in Appendix 2. The determinant of the transfer matrix is

$$\left|M_{fl}\right| = e^{-i(\omega_1 t_1 + \omega_2 t_1 + \omega_{ccl} t_2)} \tag{3.126}$$

Matrix M_{fl} can be rewrite into a complex form,

$$M_{fl} = A + iB \tag{3.127}$$

The matrix elements of A, B are given in Appendix 2. Similar with what we did in (3.83)-(3.86), the 4×4 transfer matrix for the horizontal and vertical motion of the accumulated ions can be written as

$$\begin{pmatrix} X_{cl}(t) \\ \dot{X}_{cl}(t) \\ Y_{cl}(t) \\ \dot{Y}_{cl}(t) \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & -B_{11} & -B_{12} \\ A_{21} & A_{22} & -B_{21} & -B_{22} \\ B_{11} & B_{12} & A_{11} & A_{12} \\ B_{21} & B_{22} & A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} X_{cl}(0) \\ \dot{X}_{cl}(0) \\ Y_{cl}(0) \\ \dot{Y}_{cl}(0) \end{pmatrix} = T_{fl}\begin{pmatrix} X_{cl}(0) \\ \dot{X}_{cl}(0) \\ Y_{cl}(0) \\ \dot{Y}_{cl}(0) \end{pmatrix} \tag{3.131}$$

Setting the initial condition for the ion cloud to be (1,0,0,0) and multiplying it by T_{fl} for 20 meters bunch spacing, 0.3 meters bunch length with parameter given below (3.105), the orbit of the accumulated ion can be obtained as what shown in Fig.11.1. As shown in Fig.11.1 the ion cloud motion is composed of two parts, the Larmor oscillation and the drift of the Larmor circle. In order to obtain the drift frequency, consider equation (3.101). It has the same form as (3.40) with zero Z_i to the RHS. Following the procedures from (3.42) to (3.48), the equation of motion for the Larmor circle guiding center can be derived as

$$\frac{d}{dt}\overline{z_{cl}} - \frac{i(\omega_{cle}^2 - \omega_{cli}^2)}{\omega_{ccl}}\overline{z_{cl}} = 0 \tag{3.131.1}$$

where $\overline{z_{cl}}$ describes the guiding center and is defined as

$$\overline{Z_{cl}} = \frac{1}{T_{ccl}}\int_0^{T_{ccl}} Z_{cl}(t)dt \tag{3.131.2}$$

where $T_{ccl} = \dfrac{2\pi}{\omega_{ccl}}$ is the Larmor period. The solution of (3.131.1) for each bunch period $t_1 + t_2$ is

$$\overline{z_{cl}}(t_1) = \overline{z_{cl}}(0)e^{\frac{i(\omega_{cle}^2 - \omega_{cli}^2)}{\omega_{ccl}}t_1} \tag{3.131.4}$$

where

$$t_1 = \frac{0.3}{c} = 10^{-9}s^{-1}. \tag{3.131.5}$$

From (3.103)-(3.105), one gets

$$\omega_{drift} = \frac{\omega_{cle}^2 - \omega_{cli}^2}{\omega_{ccl}} = 4.41\times 10^7 s^{-1} \tag{3.131.6}$$

The guiding center drift phase advance for each bunch period $t_1 + t_2$ is

$$\Delta\psi_{drift} = \omega_{drift}t_1 = 0.044 \text{ rad} \tag{3.131.7}$$

For one period of guiding center drift oscillation, the number of bunches needed to pass by the ion cloud is

$$N = \frac{2\pi}{\Delta\psi_{drift}} \approx 142 \tag{3.131.8}$$

This result is consistent with the turn by turn data plotting shown in Fig.11.1. Since the drift motion only happens when the bunches passing by the cloud (t_1 out of one bunch period $t_1 + t_2$), the average angular drift frequency will be given by the inverse of the time needed for one drift oscillation multiplied by 2π, i.e.

$$\Omega_{drift} = \frac{2\pi}{N(t_1 + t_2)} = 6.5 \times 10^5 s^{-1} \tag{3.131.9}$$

where

$$t_1 + t_2 = 6.77 \times 10^{-8} s \tag{3.131.10}$$

The stability condition for ion clouds motion is that the maximal amplitude of the eigenvalues of T_{fl} must be equal or smaller than 1, i.e.

$$|\lambda_{\max}| = \sqrt{\mathrm{Re}(\lambda_{\max})^2 + \mathrm{Im}(\lambda_{\max})^2} \tag{3.132}$$

Here we calculate the eigenvalues numerically and the results has been plotted in Fig.12 and Fig.13. As shown in Fig.12, for a zero magnetic field, a gap of 180 ns is enough to clear the ion clouds out of the cooling section. However, as the magnetic field increases, the stable region increases as well and when the magnetic field is around a few Tesla, it is not likely that the ion clouds can be cleared out by simply making a gap for the circulating beams. One more efficient way could be adjusting the strength of the magnetic field to the unstable region as shown in Fig. 13. For instance, when there is no magnetic field, bunch spacing of 20 meters will make the ion accumulate inside the cooling section but if a longitudinal magnetic field of 0.79-0.98 is applied, the ion clouds can be cleared out by the first resonance shown in Fig.13. It is also clear from Fig.13, a bunch spacing of 60 meters can not clear out the ion clouds if the magnetic field sitting at any region where the maximal amplitude of the eigenvalues is one. Although the electrostatic force coming from the ion clouds itself has been ignored in the above discussion, it can be included into the equation of motion (3.101) and (3.118) easily as shown in the following,

$$\frac{d^2}{dt^2} z_{cl} + i\omega_{ccl} \frac{d}{dt} z_{cl} - \omega_{pcl}^2 z_{cl} = 0 \tag{3.134}$$

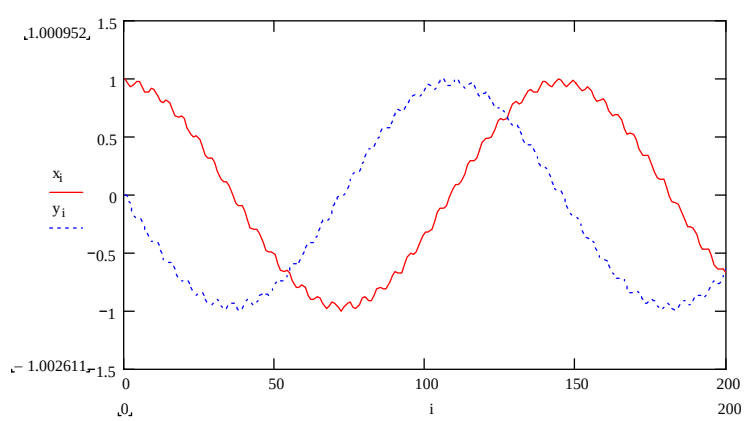


Fig.11.1 The orbit of the accumulated ion in the cooling section. The x axis shows the number of bunches passing by and the y axis shows the transverse position of the ion. The red solid curve is for the horizontal position and the blue dot curve is for the vertical position.

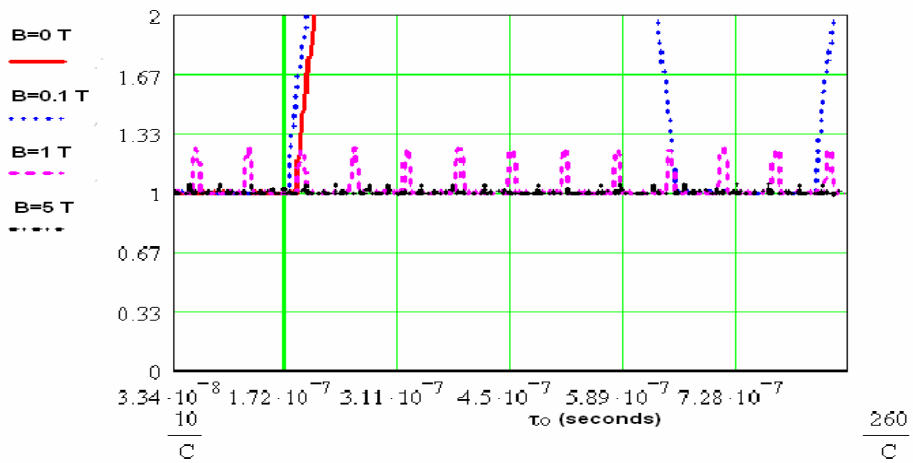


Fig. 12 The dependence of the maximal amplitude of the eigenvalues on the bunch spacing. The x axis is the spacing between two successive bunches in unit of seconds and the y axis is the amplitude of the maximal eigenvalue. Four curve are plotted for magnetic field equal to 0, 0.1,1 and 5 Tesla. The interval between two successive resonances is approximately equal to the Lamoure frequency of the ion clouds.The bunch length is taken as 0.3 meters.

$$\frac{d^2}{dt^2}z_{cl} + i\omega_{ccl}\frac{d}{dt}z_{cl} + (\omega_{cle}^2 - \omega_{cli}^2 - \omega_{pcl}^2)z_{cl} = 0 \tag{3.133}$$

where

$$\omega_{pcl} = \sqrt{\frac{Z_{cl}^2 n_{cl} e^2}{2\varepsilon_0 m_{cl}}} \tag{3.135}$$

The ion cloud density is usually expressed into the neutralization factor η defined as the following,.

$$\eta = \frac{n_{cl}}{n_e} \tag{3.136}$$

So equation (3.135) can be rewritten as

$$\omega_{pcl} = \sqrt{\frac{\eta Z_{cl}^2 n_e e^2}{2\varepsilon_0 m_{cl}}} = \sqrt{\eta Z_{cl}}\,\omega_{cle} \tag{3.137}$$

The procedures to solve equation (3.133) and (3.134) are the same as what has been done for equation (3.101) and the transfer matrix for them are:

a). For region 1, i.e. (3.133), the transfer matrix has the same form as (3.117) except the eigenfrenquecies includes the ion clouds term now

$$M_{focus}(t_1) = \frac{1}{\omega_{f,1} - \omega_{f,2}}\begin{pmatrix} \omega_{f,1}e^{-i\omega_{f,2}t_1} - \omega_2 e^{-i\omega_{f,1}t_1} & -i(e^{-i\omega_{f,2}t_1} - e^{-i\omega_{f,1}t_1}) \\ -i\omega_1\omega_2(e^{-i\omega_2 t_1} - e^{-i\omega_{f,1}t_1}) & \omega_1 e^{-i\omega_{f,1}t_1} - \omega_2 e^{-i\omega_{f,2}t_1} \end{pmatrix} \tag{3.138}$$

Fig.13 The dependence of the maximal amplitude of the eigenvalues on the field strength of the solenoid. The x axis is the longitudinal magnetic field of the solenoid in unit of Tesla and the y axis is the maximal amplitude of the eigenvalues. The red solid curve is for the bunch spacing equal to 60 meters (200 ns) and the blue dash curve is for the bunch spacing to be 20 meters (67 ns). The bunch length is taken as 0.3 meters.

where,

$$\omega_{f1,2} = \frac{1}{2}\left[\omega_{ccl} \pm \sqrt{\omega_{ccl}^2 + 4(\omega_{cle}^2 - \omega_{cli}^2 - \omega_{clcl}^2)}\right] \tag{3.139}$$

b). For region 2, i.e. (3.134), the transfer matrix also has the similar form as (3.117) instead of (3.123) but with a different eigenfrequencies.

$$M_{defocus} = \frac{1}{\omega_{d,1} - \omega_{d,2}}\begin{pmatrix} \omega_{d,1}e^{-i\omega_{d,2}t} - \omega_{d,2}e^{-i\omega_{d,1}t} & -i(e^{-i\omega_{d,2}t} - e^{-i\omega_{d,1}t}) \\ -i\omega_{d,1}\omega_{d,2}(e^{-i\omega_{d,2}t} - e^{-i\omega_{d,1}t}) & \omega_{d,1}e^{-i\omega_{d,1}t} - \omega_2 e^{-i\omega_{d,2}t} \end{pmatrix} \tag{3.140}$$

where

$$\omega_{d1,2} = \frac{1}{2}\left[\omega_{ccl} \pm \sqrt{\omega_{ccl}^2 - 4\omega_{clcl}^2}\right] \tag{3.141}$$

Thus the transfer matrix for one bunch period is

$$M_{fd} = M_{focus} M_{defocus}$$

The 4×4 transfer matrix can be obtained again as

$$T_{fd} = \begin{pmatrix} \mathrm{Re}(M_{fd}) & -\mathrm{Im}(M_{fd}) \\ \mathrm{Im}(M_{fd}) & \mathrm{Re}(M_{fd}) \end{pmatrix} \tag{3.142}$$

By plotting the maximal eigenvalue amplitude of the transfer matrix T_{fd} as the function of the neutralization factor η, a limit for the ion accumulation can be given for a stable

ion motion. Above the limit, the defocusing effects from the ion cloud itself will stop further ion accumulation. As shown in Fig. 14, the limit is around $\eta = 0.0074$ for $B_{//} = 2T$ and $\eta = 0.0011$ for $B_{//} = 5T$.

Section 3.3.2 Transverse coherent instability in the presence of the ion cloud

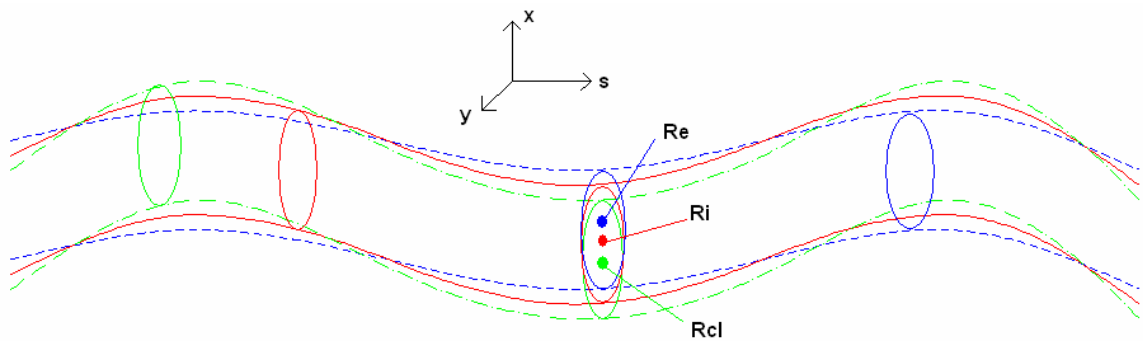


Fig.15 Illustration of the cooling section commoving beams and the ion cloud. The blue dash curve represents the electron beams, the red solid curve represents circulating ion beam and the green dot-dash curve represents the accumulated ion cloud from the residue gas ionization. The solid spots represents their centroids respectively according to the colors and the solid ellipses represent their cross section.

In section 3.2, the coherent two stream instability has been studied and a growth rate of 2.8×10^{-7} per turn has been calculated due to the dipole mode centroid oscillation. One may ask what will happen to the two stream interaction in the cooling section if the ion cloud from the ionization of the residue gas is not completely cleared out. In this section, the effects of the ion cloud to the electron-ion beam long range transverse interaction will be studied.

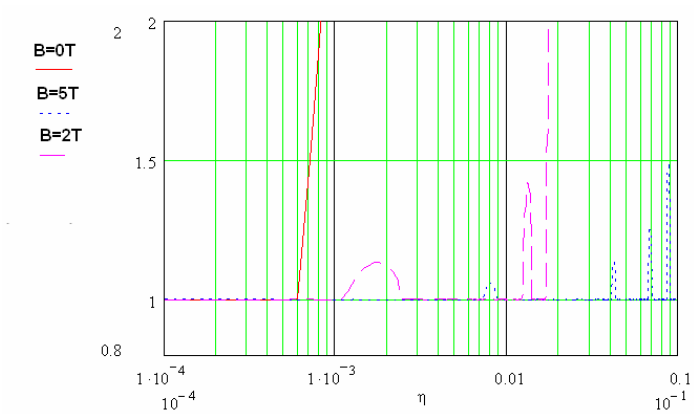


Fig. 14, The dependence of the maximal amplitude of the eigenvalues on the neutralization factor η for varies magnetic field. The x axis is the neutralization factor and the y axis is the maximal amplitude of the transfer matrix eigenvalus. The matrix is calculated from (3.143) with bunch l ength 0.3 meter and bunch spacing 20 meters (67ns).

Comparing with the situation for section 3.2, one more term due to the ion cloud has to be added into equation (3.28) and (3.29). Thus the equations of motion for a single circulating ion or a single electron in the lab frame are

$$\frac{d^2}{dt'^2}\vec{r}_e = \omega'^2_{pe}(\vec{r}_e - \vec{R}_e) - \omega'^2_{ei}(\vec{r}_e - \vec{R}_i) - \omega'_{ce}(\frac{d}{dt}\vec{r}_e \times \hat{s}) - \omega'^2_{ecl}(\vec{r}_e - \vec{R}_{cl}) \tag{3.143}$$

$$\frac{d^2}{dt'^2}\vec{r}_i = -\omega'^2_{ie}(\vec{r}_i - \vec{R}_e) + \omega'^2_{pi}(\vec{r}_i - \vec{R}_i) + \omega'_{ci}(\frac{d}{dt}\vec{r}_i \times \hat{s}) + \omega'^2_{icl}(\vec{r}_i - \vec{R}_{cl}) \tag{3.144}$$

where the subscribe 'cl' stands for 'cloud'. <u>For consistence with the previous chapter, we are going to use primed variables such as t', ω' for the quantities in lab frame and the non-primed variables such as t, ω for the quantities in the beam frame</u>**.** The equation of motion for a trapped ion is

$$\frac{d^2}{dt'^2}\vec{r}_{cl} = -\omega'^2_{cle}(\vec{r}_{cl} - \vec{R}_e) + \omega'^2_{pcl}(\vec{r}_{cl} - \vec{R}_{cl}) + \omega'_{ccl}(\frac{d}{dt'}\vec{r}_{cl} \times \hat{s}) + \omega'^2_{cli}(\vec{r}_{cl} - \vec{R}_i) - \Gamma'\frac{d}{dt'}\vec{r}_{cl} \tag{3.145}$$

where Γ' can appear, for example, because of non-linearity of "external" electrical fields created by electrons and other kinds of ions. It can be considered as free parameter.
Typically damping time is about 10-20 periods of the ion coherent perpendicular oscillations. Equation (3.143)-(3.145) are written in the lab frame and the plasma frequencies are defined as the following, (All the numbers here and later in this section are given for the lab frame densities $n'_i = 5.3\times10^{14}\,m^{-3}$, $n'_e = 3.3\times10^{17}\,m^{-3}$ and the hydrogen ion cloud, i.e. $Z_{cl} = 1$ and $m_{cl} = m_p$)

$$\omega'_{pe} = \sqrt{\frac{n'_e e^2}{2m_e\varepsilon_0\gamma^3}} = 2.295\times10^7\,s^{-1} \qquad \omega'_{pi} = \sqrt{\frac{Z_i^2 n'_i e^2}{2M_i\varepsilon_0\gamma^3}} = 1.21\times10^5\,s^{-1}$$

$$\omega'_{ie} = \sqrt{\frac{Z_i n'_e e^2}{2M_i\varepsilon_0\gamma^3}} = 3.4\times10^5\,s^{-1} \qquad \omega'_{ei} = \sqrt{\frac{Z_i n'_i e^2}{2m_e\varepsilon_0\gamma^3}} = 8.16\times10^6\,s^{-1}$$

$$\omega'_{icl} = \sqrt{\frac{Z_i Z_{cl} n'_{cl} e^2}{2M_i\varepsilon_0\gamma}} = 3.4\sqrt{\eta}\times10^7\,s^{-1} \qquad \omega'_{ecl} = \sqrt{\frac{Z_{cl} n'_{cl} e^2}{2m_e\varepsilon_0\gamma}} = 2.295\sqrt{\eta}\times10^9\,s^{-1}$$

$$\omega'_{cle} = \sqrt{\frac{Z_{cl} n'_e e^2}{2m_{cl}\varepsilon_0}} = 5.37\times10^8\,s^{-1} \qquad \omega'_{cli} = \sqrt{\frac{Z_i Z_{cl} n'_i e^2}{2m_{cl}\varepsilon_0}} = 1.91\times10^8\,s^{-1} \tag{3.146}$$

where

$$n'_e = \gamma n_e = 3.3\times10^{17}\,m^{-3}$$

$$n'_i = \gamma n_i = 5.3\times10^{14}\,m^{-3}$$

The cyclotron frequencies are defined as

$$\omega'_{ccl} = \frac{Z_{cl} e B_{//}}{m_{cl}} = 4.82 \times 10^8 s^{-1} \tag{3.147}$$

$$\omega'_{ci} = \frac{Z_i e B_{//}}{m_i \gamma} = 1.93 \times 10^6 s^{-1} \tag{3.148}$$

$$\omega'_{ce} = \frac{e B_{//}}{m_e \gamma} = 8.79 \times 10^9 s^{-1} \tag{3.149}$$

Following the procedures from (3.28) to (3.52), the equations of motion for the beam centroids R_i, R_e and the centroid of the ion cloud R_{cl} can be derived as

$$\frac{d^2}{dt'^2} Z_i + i\omega'_{ci} \frac{d}{dt'} Z_i + \omega'^2_{ie} (Z_i - \overline{Z_e}) - \omega'^2_{icl} (Z_i - Z_{cl}) = 0 \tag{3.150}$$

$$\frac{d}{dt'} \overline{Z_e} + i\Lambda'_{ei} (\overline{Z_e} - Z_i) + i\Lambda'_{ecl} (\overline{Z_e} - Z_{cl}) = 0 \tag{3.151}$$

$$\frac{d^2}{dt'^2} Z_{cl} + i(\omega'_{ccl} - i\Gamma') \frac{d}{dt'} Z_{cl} + \omega'^2_{cle} (Z_{cl} - Z_e) - \omega'^2_{cli} (Z_{cl} - Z_i) = 0 \tag{3.152}$$

where

$$\Lambda'_{ei} = \frac{\omega'^2_{ei}}{\omega'_{ce}} = 7.57 \times 10^3 s^{-1}$$

$$\Lambda'_{ecl} = \frac{\omega'^2_{ecl}}{\omega'_{ce}} = \eta \times 5.99 \times 10^8 s^{-1}$$

Set the trial solution to be

$$Z_{e,i,cl} = a_{e,i,cl} e^{-i(\omega' t' - k'_s s')} \tag{3.153}$$

Equation (3.153) is written in the lab frame and the wave number is given by the periodic condition for the ion cloud,

$$k'_s = \frac{n}{R} \tag{3.154}$$

where n is the harmonic number and R is the radius of the ring.

Inserting equation (3.153) into (3.152), one gets

$$-a_{cl}\omega'^2 + (\omega'_{ccl} - i\Gamma')\omega' a_{cl} + \omega'^2_{cle} (a_{cl} - a_e) - \omega'^2_{cli} (a_{cl} - a_i) = 0 \tag{3.155}$$

, which can be rewritten as

$$a_{cl} = T(\omega')(\omega'^2_{cle} a_e - \omega'^2_{cli} a_i) \tag{3.156}$$

where

$$T(\omega') = \frac{-1}{(\omega' - \Omega'_1)(\omega' - \Omega'_2)} \tag{3.157}$$

and the resonant frequencies $\Omega'_{1,2}$ are defined as

$$\Omega'_{1,2} = -\frac{i}{2}\Gamma' + \frac{1}{2}\left[\omega'_{ccl} \pm \sqrt{(\omega'_{ccl} - i\Gamma')^2 + 4(\omega'^2_{cle} - \omega'^2_{cli})}\right] \tag{3.158}$$

Inserting (3.153) into equation (3.150), (3.151) and using (3.152.1), one gets

$$(-\omega^2)a_i + \omega\omega'_{ci}\, a_i + \omega'^2_{ie}(a_i - a_e) - \omega'^2_{icl}(a_i - a_{cl}) = 0 \tag{3.159}$$

$$(-i\omega)a_e + i\Lambda'_{ei}(a_e - a_i) + i\Lambda'_{ecl}(a_e - a_{cl}) = 0 \tag{3.160}$$

where

$$\omega = \omega' - ck'_s \tag{3.161}$$

Inserting (3.156) into (3.159) and (3.160), one gets

$$\left[\omega^2 - \omega\omega'_{ci} + \omega'^2_{icl} - \omega'^2_{ie} + \omega'^2_{icl}\,\omega'^2_{cli}\,T(\omega')\right]a_i + \left[\omega'^2_{ie} - \omega'^2_{icl}\,\omega'^2_{cle}\,T(\omega')a_e\right]a_e = 0 \tag{3.162}$$

$$\left[\Lambda'_{ecl}\,T(\omega')\omega'^2_{cli} - \Lambda'_{ei}\right]a_i + \left[\Lambda'_{ecl} - \Lambda'_{ecl}\,T(\omega')\omega'^2_{cle} - \omega + \Lambda'_{ei}\right]a_e = 0 \tag{3.163}$$

For non-zero solution, the determinant of the coefficient matrix must be zero, which gives the dispersion equation, which can be solved numerically for three eigenfrequencies and the solution of (3.150) , (3.151) can be written as

$$Z_i = \sum_{\alpha=1}^{3} a_{i,\alpha} e^{-i\omega_\alpha t'} \tag{3.165}$$

$$\overline{Z}_e = \sum_{\alpha=1}^{3} a_{e,\alpha} e^{-i\omega_\alpha t'} \tag{3.166}$$

For $\eta = 10^{-4}$, $\Gamma' = 0.1 \times \mathrm{Re}(\Omega'_1)$, as an example, the eigenfrequencies are

$$\omega = \begin{cases} 1.9\times10^6 - i3.6\times10^4 s^{-1} \\ 6.4\times10^4 - i2.3\times10^5 s^{-1} \\ \qquad 7.6\times10^3 s^{-1} \end{cases} \tag{3.167}$$

Equation (3.165) and (3.166) has the same form as (3.61) and (3.62). Following the same procedures from (3.63) to (3.97), the increments of the eigenvalues can be calculated for certain ion cloud damping rate Γ' and neutralization factor η.

Fig.16 shows the calculation results of the instability increments for different neutralization level. From Fig.14, the threshold of the neutralization lever for an unstable ion clouds transverse motion is about 7×10^{-3}, which corresponding an increment of 0.01 per revolution. The neutralization level is also limited by the vaccum quality and the geometry of the cooling section.

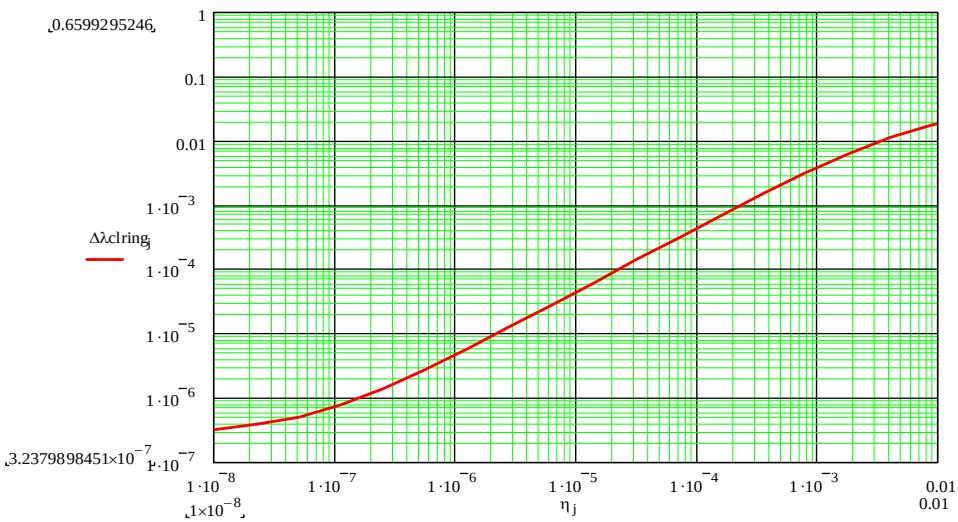
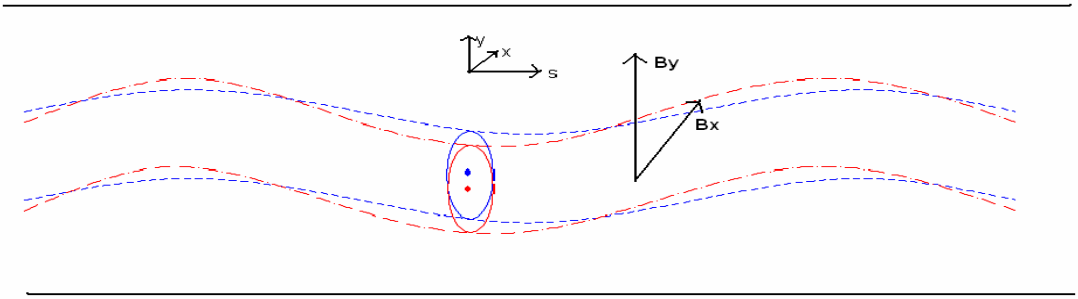


Fig.16 plots the eigenvalue increment as a function of the neutralization factor for $\Gamma' = 0.1 \times \mathrm{Re}(\Omega'_1)$ and parameters above (3.146), which shows the coherent instability strongly depends on the neutralization factor.

Section 3.4 Coherent instability in the presence of wiggler

Section 3.4.1 Transversal dipole coherent instability



For the wiggler field, the magnetic field in the lab frame is

$$B_s = 0 \tag{3.168}$$

$$B_x = B_\perp \cos(\frac{2\pi s}{\lambda_w}) \tag{3.169}$$

$$B_y = B_\perp \sin(\frac{2\pi s}{\lambda_w}) \tag{3.170}$$

where $B_\perp$, λ_w are the magnitude and the wavelength of the wiggler field respectively. Equation of motion for a single electron in cooling section with Wiggler field

$$\frac{d^2}{dt'^2}\vec{r}_e = \omega'^2_{pe}(\vec{r}_e - \vec{R}_e) - \omega'^2_{ei}(\vec{r}_e - \vec{R}_i) - \frac{e\beta c\vec{s} \times \vec{B}_\perp}{m_e \gamma} \tag{3.171}$$

The centroids equations of motion in x plane are

$$\frac{d^2}{ds^2}X_e = -\Omega'^2_{ei}(X_e - X_i) + \frac{eB_\perp}{m_e\gamma c}\sin(\Omega'_w s) \tag{3.172}$$

$$\frac{d^2}{ds^2}X_i = -\Omega'^2_{ie}(X_i - X_e) - \frac{ZeB_\perp}{M_i\gamma c}\sin(\Omega'_w s) \tag{3.173}$$

where

$$\Omega'_{ie} = \frac{\omega'_{ie}}{c}$$

$$\Omega'_{ei} = \frac{\omega'_{ei}}{c}$$

$$\Omega'_w = \frac{2\pi}{\lambda_w}$$

Subtract (3.173) by (3.172),

$$\frac{d^2}{ds^2}X_{ie} + \Omega'^2_0 X_{ie} = \hat{f}\sin(\Omega'_w s) \tag{3.174}$$

where

$$\Omega'^2_0 = \Omega'^2_{ie} + \Omega'^2_{ei}$$

$$\hat{f} = -\frac{eB_\perp}{m_e\gamma c}(1 + \frac{Zm_e}{M_i})$$

$$X_{ie} = X_i - X_e$$

Equation (3) is a forced hamornic oscillator and makes the trial solution to be

$$X_{ie} = A\sin(\Omega'_w s) \tag{3.175}$$

and inserting (3.175) into (3.174), one gets

$$A = \frac{\hat{f}}{\Omega'^2_0 - \Omega'^2_w}$$

Thus one of the particular solutions of (3) is

$$X^p_{ie} = \frac{\hat{f}}{\Omega'^2_0 - \Omega'^2_w}\sin(\Omega'_w s)$$

The general solution of the homogenous equation of (3.174) is

$$X^g_{ie} = A(z)\cos(\Omega'_0 s + \varphi(z))$$

Thus the overall solution of (3.174) is

$$\begin{aligned} X_{ie} &= X^g_{ie} + X^p_{ie} \\ &= A(z)\cos(\Omega'_0 s + \varphi(z)) + \frac{\hat{f}}{\Omega'^2_0 - \Omega'^2_w}\sin(\Omega'_w s) \end{aligned} \tag{3.175}$$

Set the initial condition to be:

$$X_e(z,0) = \frac{d}{dt}X_e(z,0) = 0 \tag{3.176}$$

Thus the equation of motion for the ion center is

$$\frac{d^2}{ds^2}X_i = -\Omega'^2_{ie}\left[\cos(\Omega'_0 s)X_i(0) + \frac{\sin(\Omega'_0 s)}{\Omega'_0}(X'_i(z,0) - \frac{\hat{f}}{\Omega'^2_0 - \Omega'^2_w}\Omega'_w)\right] - \left[\frac{\Omega'^2_{ie}}{\Omega'^2_0 - \Omega'^2_w}\hat{f} + \frac{ZeB_\perp}{M_i\gamma c}\right]\sin(\Omega'_w s) \quad (3.177)$$

Integrate equation (3.177), one gets

$$\begin{pmatrix} X_i \\ X'_i \\ 1 \end{pmatrix}_s = M_{transverse}\begin{pmatrix} X_i \\ X'_i \\ 1 \end{pmatrix}_0 \quad (3.178)$$

where

$$M_{transverse} = \begin{pmatrix} 1+\xi'(\cos(\Omega'_0 s)-1) & \frac{1}{\Omega'_0}\left[\Omega'_0 s(1-\xi') + \xi'\sin(\Omega'_0 s)\right] & a(s) \\ -\xi'\Omega'_0\sin(\Omega'_0 s) & 1+\xi'(\cos(\Omega'_0 s)-1) & a'(s) \\ 0 & 0 & 1 \end{pmatrix} \quad (3.179)$$

and

$$a(s) = -\frac{\hat{f}\xi'\xi'_w}{1-\xi'_w}\frac{(\sin(\Omega'_0 s) - \Omega'_0 s)}{\Omega'_0\Omega'_w} - \left[\frac{\xi'}{1-\xi'_w}\hat{f} + \frac{ZeB_\perp}{M_i\gamma c}\right]\frac{(\sin(\Omega'_w s) - \Omega'_w s)}{\Omega'^2_w} \quad (3.180)$$

$$\xi'_w = \frac{\Omega'^2_w}{\Omega'^2_0}$$

$$\xi' = \frac{\Omega'^2_{ie}}{\Omega'^2_0}$$

The transfer matrix for the ring is

$$M_{ring} = R_x L_{drift} M_{transverse} L_{drift} \quad (3.181)$$

where

$$R_x = \begin{pmatrix} \cos(2\pi\nu_x) & \beta_x\sin(2\pi\nu_x) & 0 \\ \frac{1}{\beta_x}\sin(2\pi\nu_x) & \cos(2\pi\nu_x) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.182)$$

$$L_{drift} = \begin{pmatrix} 1 & -\frac{L}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.183)$$

For the parameters listed in table 3, the transfer matrix, its determinant increment and the eigenvalue increment for the ring are[1]

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[1] The plasma frequencies and the wiggler frequency for the parameters listed in Table 3.are

$\Omega'_{ie} = 3.41\times10^{-4} rad/m$ $\Omega'_{ei} = 5.68\times10^{-3} rad/m$ $\Omega'_0 = 5.69\times10^{-3} rad/m$ $\Omega'_w = 41.89 rad/m$ $\hat{f} = -0.0586 m/m^2$

$\xi' = 3.58\times10^{-3}$ $\xi'_w = 5.43\times10^{7}$ $a(60) = 2.44\times10^{-6} m$ $a'(100) = 2.89\times10^{-8} m/m$

Wiggler field strength	$0.001\ T$
Wiggler field wavelength	$0.15\,m$
Cooling section length	$60\,m$
Electron beam size	$2.36\ mm$
Electron rms bunch length	$9\,mm$
Electron beam charge	$5\,nC$
Electron Density	$2.96\times10^{16}\ m^{-3}$
Ion beam horizontal tune	28.23
Ion beam vertical tune	29.23
Ion beam charge	$12.64\ nC$
Ion rms bunch length	$0.37\,m$
Ion beam size	$2.36\ mm$
Ion beam density	$2.31\times10^{13}\ m^{-3}$

Table 3. The parameters for the current non-magnetized electron cooler design.

$$M_{ring} = \begin{pmatrix} 0.109 & 0.993 & 2.004\times10^{-7} \\ -0.995 & 0.107 & -1.562\times10^{-6} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left|\lambda_{transverse}\right| - 1 = -2.0\times10^{-6} \tag{3.184}$$

$$\left|M_{transverse}\right| - 1 = -4.0\times10^{-6} \tag{3.185}$$

To obtain the threshold for the instability, the determinant and eigenvalue increment are plotted as a function of the electron beam density in Fig.17, which is the same as the straight section case. Thus the threshold for transverse dipole instability is

$$n_e^{th} = 2.99\times10^{19}\,m^{-3} \tag{3.186}$$

which is three orders of magnitude larger than the current electron density.

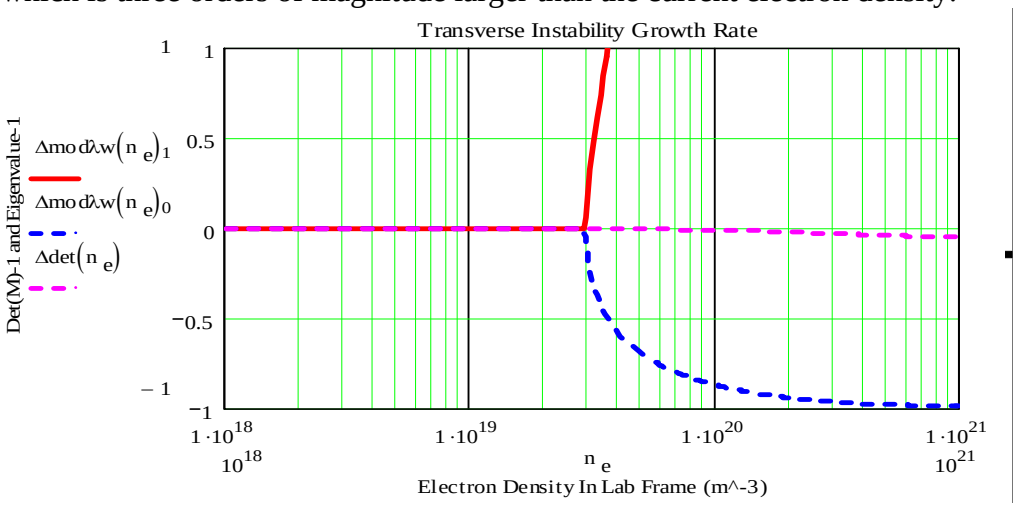

Fig.17 The increment of the determinant and the eigenvalues of the transversal transfer matrix for the ring.

Section 3.4.2 Longitudinal dipole coherent instability

Since the wiggler field does not affect the longitudinal motion, the threshold for the instability due to Langmuir oscillation can still be determined from equation (2.19) and (2.22).

As shown in Fig.18, the threshold for the determinant less than one is reduced to

$$n_e^{thd} = 3.1 \times 10^{17} m^{-3} \tag{3.187}$$

and for the eigenvalue less than one is reduced to

$$n_e^{th\lambda} = 7.7 \times 10^{16} m^{-3} \tag{3.188}$$

After including the rest of the RHIC ring, the threshold from the eigenvalue is the same as what from the determinant limitation, which is shown in Fig.18 and described in Section 2.4. Since the cooling section is much longer than the magnetized cooling design, the maximal growth rate is bigger than the synchrotron tune and the instability could happen before the synchrotron motion distorts the longitudinal plasma oscillation. The electron bunch length threshold can be estimated from equation (3.187) and Table 3 +

to be

$$l_e^{th} = 1.1cm$$

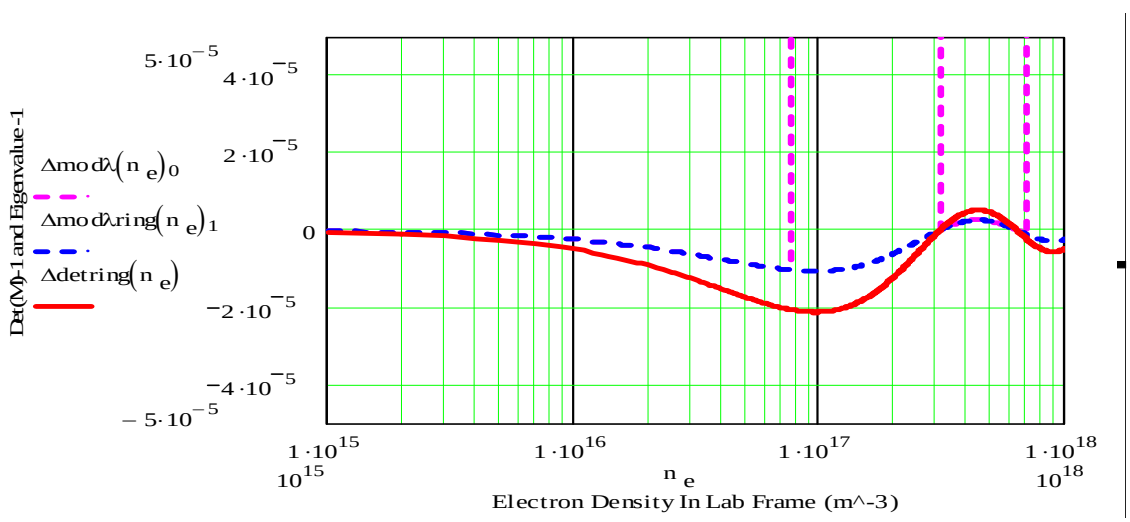

Fig. 18 The increments of the determinant and the eigenvalue of the longitudinal transfer Marix for parameters listed in table 3. The x axis is the electron density in beam frame in units of m^{-3} the light blue line shows the synchrotron tune, 3.7×10^{-4} and other curves have the same definition as Fig.4.1.

Appendix1: Equation of motion for single particle

In this section, we get the solution for the ion beam centroid motion in the cooling section due to its interaction with electron beam. However it will be necessary to know the behavior of each single ion in the ion beam for the purpose of simulation. The equation of motion for single ion particle can be derived as the following. Consider equation (3.9)

$$\frac{d^2}{dt^2}\vec{r}_i = (\omega_{pi}^2 - \omega_{ie}^2)\vec{r}_i + \omega_{ie}^2\vec{R}_e - \omega_{pi}^2\vec{R}_i \qquad (3.25.1)$$

Since we already derived the solution for $\vec{R}_i$, we only need to find the behavior of the electron centroid to get a explicit form of the single ion equation of motion. From (3.13)

$$\vec{R}_i - \vec{R}_e = \vec{R}_i(z,0)\cos(\omega_0 t) + \frac{\sin(\omega_0 t)}{\omega_0}\dot{\vec{R}}_i(z,0) \qquad (3.25.2)$$

Thus, we get the solution for $\vec{R}_e$

$$\vec{R}_e = \vec{R}_i - \vec{R}_i(z,0)\cos(\omega_0 t) - \frac{\sin(\omega_0 t)}{\omega_0}\dot{\vec{R}}_i(z,0) \qquad (3.25.3)$$

Insert (3.25.3) into (3.25.1)

$$\frac{d^2}{dt^2}\vec{r}_i = (\omega_{pi}^2 - \omega_{ie}^2)\vec{r}_i + (\omega_{ie}^2 - \omega_{pi}^2)\vec{R}_i - \omega_{ie}^2\left[\vec{R}_i(z,0)\cos(\omega_0 t) + \frac{\sin(\omega_0 t)}{\omega_0}\dot{\vec{R}}_i(z,0)\right] \qquad (3.25.4)$$

Insert equation (3.20) into (3.25.4)

$$\begin{aligned}\frac{d^2}{dt^2}\vec{r}_i - (\omega_{pi}^2 - \omega_{ie}^2)\vec{r}_i = &\left(\xi(\omega_{ie}^2 - \omega_{pi}^2) - \omega_{ie}^2\right)\left[\frac{\dot{\vec{R}}_i(z,0)}{\omega_0}\sin(\omega_0 t) + \vec{R}_i(z,0)\cos(\omega_0 t)\right] \\ &+ (1-\xi)(\omega_{ie}^2 - \omega_{pi}^2)\left[\dot{\vec{R}}_i(z,0)t + \vec{R}_i(z,0)\right]\end{aligned} \qquad (3.25.5)$$

Thus, the single ion equation of motion inside the solenoid is like a driving oscillator

$$\frac{d^2}{dt^2}\vec{r}_i - (\omega_{pi}^2 - \omega_{ie}^2)\vec{r}_i = f(t) \qquad (3.25.6)$$

where $f(t)$ is the driving force and given by

$$f(t) = (\xi(\omega_{ie}^2 - \omega_{pi}^2) - \omega_{ie}^2)\left[\frac{\dot{\vec{R}}_i(z,0)}{\omega_0}\sin(\omega_0 t) + \vec{R}_i(z,0)\cos(\omega_0 t)\right] + (1-\xi)(\omega_{ie}^2 - \omega_{pi}^2)\left[\dot{\vec{R}}_i(z,0)t + \vec{R}_i(z,0)\right] \qquad (3.25.7)$$

Since for our case, ξ is small,

$$f(t) \approx -\omega_{ie}^2 \left[\frac{\dot{\vec{R}}_i(z,0)}{\omega_0} \sin(\omega_0 t) + \vec{R}_i(z,0)\cos(\omega_0 t) \right] + (\omega_{ie}^2 - \omega_{pi}^2)\left[\dot{\vec{R}}_i(z,0)t + \vec{R}_i(z,0)\right] \qquad (3.25.8)$$

In the presence of a solenoid, the single ion motion can be described by equation (3.29) or

$$\frac{d^2}{dt^2}\vec{r}_i = -\omega_{ie}^2(\vec{r}_i - \vec{R}_e) + \omega_{pi}^2(\vec{r}_i - \vec{R}_i) + \omega_{ci}(\frac{d}{dt}\vec{r}_i \times \hat{s}) \qquad (3.100.1)$$

Write the above equation into the vertical and horizontal plane,

$$\frac{d^2}{dt^2}x_i = -\omega_{ie}^2(x_i - X_e) + \omega_{pi}^2(x_i - X_i) + \omega_{ci}\frac{d}{dt}y_i \qquad (3.100.2)$$

$$\frac{d^2}{dt^2}y_i = -\omega_{ie}^2(y_i - Y_e) + \omega_{pi}^2(y_i - Y_i) - \omega_{ci}\frac{d}{dt}x_i \qquad (3.100.3)$$

$(3.100.2) + i(3.100.3) \Rightarrow$

$$\frac{d^2}{dt^2}z_i = -\omega_{ie}^2(z_i - \overline{Z}_e) + \omega_{pi}^2(z_i - Z_i) - i\omega_{ci}\frac{d}{dt}z_i \qquad (3.100.4)$$

Here z_i describes the single ion position and defined as

$$z_i = x_i + iy_i \qquad (3.100.5)$$

Equation (3.100.4) can be rewritten as

$$\frac{d^2}{dt^2}z_i + i\omega_{ci}\frac{d}{dt}z_i - (\omega_{pi}^2 - \omega_{ie}^2)z_i = \omega_{ie}^2\overline{Z}_e - \omega_{pi}^2 Z_i \qquad (3.100.6)$$

From (3.73),

$$Z_i(t) = m_{11}Z_i(0) + m_{12}\dot{Z}_i(0) \qquad (3.100.7)$$

$$\overline{Z}_e(t) = m_{31}Z_i(0) + m_{32}\dot{Z}_i(0) \qquad (3.100.8)$$

where m_{11}, m_{12}, m_{31} and m_{32} are defined in (3.76)-(3.79.2)

Thus the equation of motion for a single ion is

$$\frac{d^2}{dt^2}z_i + i\omega_{ci}\frac{d}{dt}z_i - (\omega_{pi}^2 - \omega_{ie}^2)z_i = f(t) \qquad (3.100.9)$$

where the driving force $f(t)$ is defined as the following

$$f(t) = (\omega_{ie}^2 m_{31} - \omega_{pi}^2 m_{11})Z_i(0) + (\omega_{ie}^2 m_{32} - \omega_{pi}^2 m_{12})\dot{Z}_i(0) \qquad (3.100.10)$$

We obtained the equation of motion of single ion and by numerically integrating the equation of motion, the beam behavior can be predicted.

Appendix2:Transverse Transfer Matrix Elements Definition

The matrix elements of the complex matrix M_{ion} can be obtained from (3.72) as the following,

$$m_{11} = \frac{\omega_2\left(e^{-i\omega_1 t} - T_1\right) - \omega_1\left(e^{-i\omega_2 t} - T_2\right)}{\omega_2\left(1 - T_1\right) - \omega_1\left(1 - T_2\right)} \tag{3.76}$$

$$m_{12} = \frac{i\left(T_2 - 1\right)e^{-i\omega_1 t} - i\left(T_1 - 1\right)e^{-i\omega_2 t} + i(T_1 - T_2)}{\omega_2\left(1 - T_1\right) - \omega_1\left(1 - T_2\right)} \tag{3.77}$$

$$m_{21} = \frac{i\omega_1\omega_2\left(e^{-i\omega_2 t} - e^{-i\omega_1 t}\right)}{\omega_2\left(1 - T_1\right) - \omega_1\left(1 - T_2\right)} \tag{3.78}$$

$$m_{22} = \frac{\omega_2 e^{-i\omega_2 t}\left(1 - T_1\right) - \omega_1 e^{-i\omega_1 t}\left(1 - T_2\right)}{\omega_2\left(1 - T_1\right) - \omega_1\left(1 - T_2\right)} \tag{3.79}$$

$$m_{31} = \frac{T_1\omega_2(e^{-i\omega_1 t} - 1) - T_2\omega_1(e^{-i\omega_2 t} - 1)}{\omega_2\left(1 - T_1\right) - \omega_1\left(1 - T_2\right)} \tag{3.79.1}$$

$$m_{32} = \frac{iT_1(T_2 - 1)e^{-i\omega_1 t} - iT_2(T_1 - 1)e^{-i\omega_2 t} + i(T_1 - T_2)}{\omega_2\left(1 - T_1\right) - \omega_1\left(1 - T_2\right)} \tag{3.79.2}$$

The matrix elements for the corresponding real and imaginary part of M_{ion} $A_{i,j}$ and $B_{i,j}$ are

$$A_{11} = \frac{\omega_2\left(\cos(\omega_1 t) - T_1\right) - \omega_1\left(\cos(\omega_2 t) - T_2\right)}{\omega_2\left(1 - T_1\right) - \omega_1\left(1 - T_2\right)}$$

$$A_{12} = \frac{\left(T_2 - 1\right)\sin(\omega_1 t) - \left(T_1 - 1\right)\sin(\omega_2 t)}{\omega_2\left(1 - T_1\right) - \omega_1\left(1 - T_2\right)}$$

$$A_{21} = \frac{\omega_1\omega_2\left(\sin(\omega_2 t) - \sin(\omega_1 t)\right)}{\omega_2\left(1 - T_1\right) - \omega_1\left(1 - T_2\right)}$$

$$A_{22} = \frac{\omega_2\left(1 - T_1\right)\cos(\omega_2 t) - \omega_1\left(1 - T_2\right)\cos(\omega_1 t)}{\omega_2\left(1 - T_1\right) - \omega_1\left(1 - T_2\right)}$$

$$B_{11} = -\frac{\omega_2\left(\sin(\omega_1 t) - T_1\right) - \omega_1\left(\sin(\omega_2 t) - T_2\right)}{\omega_2\left(1 - T_1\right) - \omega_1\left(1 - T_2\right)}$$

$$B_{12} = \frac{\left(T_2 - 1\right)\cos(\omega_1 t) - \left(T_1 - 1\right)\cos(\omega_2 t) + (T_1 - T_2)}{\omega_2\left(1 - T_1\right) - \omega_1\left(1 - T_2\right)}$$

$$B_{21} = \frac{\omega_1\omega_2\left(\cos(\omega_2 t) - \cos(\omega_1 t)\right)}{\omega_2\left(1-T_1\right) - \omega_1\left(1-T_2\right)}$$

$$B_{22} = \frac{\omega_1\left(1-T_2\right)\sin(\omega_1 t) - \omega_2\left(1-T_1\right)\sin(\omega_2 t)}{\omega_2\left(1-T_1\right) - \omega_1\left(1-T_2\right)}$$

The transfer matrix for the ion clouds motion within the solenoid, M_{fl}, has the following elements

$$m_{11} = \frac{\omega_1 e^{-i\omega_2 t_1} - \omega_2 e^{-i\omega_1 t_1}}{\omega_1 - \omega_2} \tag{3.125}$$

$$m_{12} = \frac{1}{\omega_1 - \omega_2}\left[i(\frac{\omega_1}{\omega_{ccl}} - 1)e^{-i(\omega_2 t_1 + \omega_{ccl} t_2)} - i(\frac{\omega_2}{\omega_{ccl}} - 1)e^{-i(\omega_1 t_1 + \omega_{ccl} t_2)} - i\frac{1}{\omega_{ccl}}(\omega_1 e^{-i\omega_2 t_1} - \omega_2 e^{-i\omega_1 t_1})\right]$$

$$m_{21} = \frac{-i\omega_1\omega_2(e^{-i\omega_2 t_1} - e^{-i\omega_1 t_1})}{\omega_1 - \omega_2}$$

$$m_{22} = \frac{1}{\omega_1 - \omega_2}\left[\frac{\omega_2(\omega_1 - \omega_{ccl})}{\omega_{ccl}} e^{-i(\omega_2 t_1 + \omega_{ccl} t_2)} - \frac{\omega_1(\omega_2 - \omega_{ccl})}{\omega_{ccl}} e^{-i(\omega_1 t_1 + \omega_{ccl} t_2)} - \frac{\omega_1\omega_2}{\omega_{ccl}}(e^{-i\omega_2 t_1} - e^{-i\omega_1 t_1})\right]$$

Matrix M_{fl} can be rewrite into a complex form,

$$M_{fl} = A + iB \tag{3.127}$$

where,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{3.128}$$

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \tag{3.129}$$

and the matrix elements are given as the following,

$$A_{11} = \frac{\omega_1 \cos(\omega_2 t_1) - \omega_2 \cos(\omega_1 t_1)}{\omega_1 - \omega_2}$$

$$B_{11} = \frac{\omega_2 \sin(\omega_1 t_1) - \omega_1 \sin(\omega_2 t_1)}{\omega_1 - \omega_2}$$

$$A_{12} = \frac{(\omega_1 - \omega_{ccl})\sin(\omega_2 t_1 + \omega_{ccl} t_2) - (\omega_2 - \omega_{ccl})\sin(\omega_1 t_1 + \omega_{ccl} t_2) - \omega_1 \sin(\omega_2 t_1) + \omega_2 \sin(\omega_1 t_1)}{\omega_{ccl}(\omega_1 - \omega_2)}$$

$$B_{12} = \frac{(\omega_1 - \omega_{ccl})\cos(\omega_2 t_1 + \omega_{ccl} t_2) - (\omega_2 - \omega_{ccl})\cos(\omega_1 t_1 + \omega_{ccl} t_2) - \omega_1 \cos(\omega_2 t_1) + \omega_2 \cos(\omega_1 t_1)}{\omega_{ccl}(\omega_1 - \omega_2)}$$

$$A_{21} = \frac{\omega_1 \omega_2}{\omega_1 - \omega_2}(\sin(\omega_1 t_1) - \sin(\omega_2 t_1))$$

$$B_{21} = \frac{\omega_1 \omega_2}{\omega_1 - \omega_2}(\cos(\omega_1 t_1) - \cos(\omega_2 t_1))$$

$$A_{22} = \frac{\omega_2(\omega_1 - \omega_{ccl})\cos(\omega_2 t_1 + \omega_{ccl} t_2) - \omega_1(\omega_2 - \omega_{ccl})\cos(\omega_1 t_1 + \omega_{ccl} t_2) - \omega_1 \omega_2(\cos(\omega_2 t_1) - \cos(\omega_1 t_1))}{\omega_{ccl}(\omega_1 - \omega_2)}$$

$$B_{22} = \frac{-\omega_2(\omega_1 - \omega_{ccl})\sin(\omega_2 t_1 + \omega_{ccl} t_2) + \omega_1(\omega_2 - \omega_{ccl})\sin(\omega_1 t_1 + \omega_{ccl} t_2) + \omega_1 \omega_2(\sin(\omega_2 t_1) - \sin(\omega_1 t_1))}{\omega_{ccl}(\omega_1 - \omega_2)}$$

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