Linear Electron Response to An Ion (Analytical Approach)

Michael Blaskiewicz
Gang Wang

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Outline

• Introduction
• Assumptions and Equation of motion
• Analytical Solution For $f_0^e(\vec{v})$=Lorenzian Distribution
  ▪ Space domain solution.
  ▪ Wavelength domain solution (It shows how electron response depends on wavelength).
• Numerical Solution For $f_0^e(\vec{v})$=Gaussian Distribution
  ▪ Checking codes with analytical formula for Lorenzian distribution (Debug).
  ▪ Comparing results for Gaussian and Lorenzian distribution.
Introduction

• The response of the electron beam to moving ions is important for CEC. The linear response will be amplified to cool the ions and the non-linear response might set limitation to the cooling rate.

• Under certain assumptions, analytical formula can be found to describe the linear response. Numerical solution is relatively straightforward for more general case.
Assumptions

• The electron beam is treated as an infinite single-species plasma with 3D temperatures.

• The velocity distribution of the electron beam is Lorenzian. (Numerical solution can be found for other velocity distributions.)

• The response has to be weak compared with the background in order to stay in linear region.
Equation Of Motion

- Linearized Vlasov Equation

\[
\frac{\partial}{\partial t} f_1(\bar{x}, \bar{v}, t) + \bar{v} \cdot \frac{\partial}{\partial \bar{x}} f_1(\bar{x}, \bar{v}, t) - \frac{eE}{m_e} \frac{\partial}{\partial \bar{v}} f_0(\bar{v}) = 0
\]

\[
\vec{\nabla} \cdot \vec{E} = \frac{\rho(\bar{x}, t)}{\varepsilon_o}
\]

\[
\rho(\bar{x}, t) = eZ \delta(\bar{x} - \bar{v}_0 t) - e \int f_1(\bar{x}, \bar{v}, t) d\bar{v}
\]

Defined as \( \tilde{n}_1(\bar{x}, t) \)
Equation Of Motion Contin.

• Fourier transform the equation of motion to k space.

\[
\frac{\partial}{\partial t} f_1(\vec{k}, \vec{v}, t) + i \vec{k} \cdot \vec{v} f_1(\vec{x}, \vec{v}, t) + i \frac{e}{m_e} \Phi(\vec{k}, t) \vec{k} \cdot \frac{\partial}{\partial \vec{v}} f_0(\vec{v}) = 0
\]

\[
\Phi(\vec{k}, t) = \frac{e}{\varepsilon_0 k^2} \left[ Z_i - \tilde{n}_1(\vec{k}, t) \right]
\]

• Integrating the equation over \( \vec{v} \), one gets

\[
\tilde{n}_1(\vec{k}, t) = \omega^2_p \int_0^t \left[ \tilde{n}_1(\vec{k}, t_1) - Z_i \right] (t_1 - t) g(\vec{k}(t_1 - t)) dt_1
\]

\[
g(\vec{u}) = \int f_0(\vec{v}) e^{i\vec{u} \cdot (\vec{v} + \vec{v}_0)} d^3 \vec{v}
\]

Ion velocity
Solution For Lorenzian Distribution

- Equilibrium electron velocity distribution

\[ f_0(v) = \frac{1}{\pi^2 \sigma_x \sigma_y \sigma_z} \left( \frac{1}{1 + \frac{v_x^2}{\sigma_x^2} + \frac{v_y^2}{\sigma_y^2} + \frac{v_z^2}{\sigma_z^2}} \right)^2 \]

which gives

\[ g(u) = \exp(iu \cdot \tilde{v}_0) \exp\left(-\sqrt{u_x \sigma_x}^2 + (u_y \sigma_y)^2 + (u_z \sigma_z)^2\right) \]

- For Lorenzian distribution, the integral equation reduced to an 2\textsuperscript{nd} order ODE

\[ \ddot{H}(\vec{k}, t) = -\omega_p^2 H(\vec{k}, t) + Z_i \omega_p^2 e^{-\lambda(\vec{k}) t} \]

\[ H(\vec{k}, t) \equiv \tilde{n}_1(\vec{k}, t) \exp\left[-\lambda(\vec{k}) \cdot t\right] \]
Solution in K space

where

\[ \lambda(\vec{k}) = ik \cdot \vec{v}_0 - \sqrt{(k_x \sigma_x)^2 + (k_y \sigma_y)^2 + (k_z \sigma_z)^2} \]

- The electron response to a certain wavelength is

\[ \tilde{n}_1(\vec{k}, t) = \frac{\omega_p^2 Z_i}{\omega_p^2 + \lambda(\vec{k})^2} \left\{ 1 - \exp(\lambda(\vec{k}) \cdot t) \cdot \left[ \cos(\omega_p t) - \frac{\lambda(\vec{k})}{\omega_p} \sin(\omega_p t) \right] \right\} \]

- The response drops for large wavelength.
- The Landau damping rate increases with wavelength and temperature.
Wavelength Depend. of the Electron Response

The wave vector $k$ is along the moving direction of the ion.

$$\frac{\vec{k} \cdot \vec{v}}{kv} = 1$$
Solution in x space

The solution in x space can be obtained by Fourier transform.

\[
\tilde{n}_1(\tilde{x}, t) = \frac{Z_i \omega_p t \sin(\omega_p t)}{\pi^2 \sigma_x \sigma_y \sigma_z \left(t^2 + \frac{(x + v_{0x} t)^2}{\sigma_x^2} + \frac{(y + v_{0y} t)^2}{\sigma_y^2} + \frac{(z + v_{0z} t)^2}{\sigma_z^2}\right)^2}
\]

If one uses the normalized variable

\[\psi \equiv \omega_p t\]

\[\tilde{v}_{0i} = \frac{\vec{v}_{0i}}{\sigma_i}\]

\[\bar{x}_i = \frac{x_i}{r_{Di}}\]

the electron density fluctuation is

\[
\tilde{n}_1(\bar{x}, t) = \frac{Z_i}{\pi^2 r_{Di} r_{Dy} r_{Dz}} \int_0^\psi \psi_1 \sin(\psi_1) d\psi_1
\]

\[
\left(\psi_1^2 + (\bar{x} + \bar{v}_{0x} \psi_1)^2 + (\bar{v} + \bar{v}_{0y} \psi_1)^2 + (\bar{z} + \bar{v}_{0z} \psi_1)^2\right)^2
\]

which can be expressed into sum of sine integral.
Density Response for an Rest Ion

- The solution reduced to the well known Debye screening formula at the condition

\[ t \to \infty \quad \vec{v}_0 = 0 \quad r_{Dx} = r_{Dy} = r_{Dz} \]

\[ \tilde{n}_1(\vec{x}, t) = \frac{Z_i}{\pi^2 r_D^3} \int_0^\infty \frac{\psi \sin(\psi) d\psi}{(\psi^2 + \vec{r}^2)^2} \]

\[ = \frac{Z_i}{4\pi r_D^2} \cdot \frac{1}{r} \exp \left( -\frac{r}{r_D} \right) \]
Density Response for an moving Ion

- The formula can describe the dynamics of the electron shielding for an ion with random 3D velocity in an infinite plasma (isotropic or anisotropic).

\[ n_{2z1} \quad \psi = \pi \quad \bar{v}_0 = (0,0,10) \]
Time Dependence of Electron Response

\[ x = y = 0.4r_D \]
Numerical Solution
(Gaussian Distribution)

• For general electron velocity distribution $f_0(\vec{v})$, the solution can be found by numerically solving the integral equation in $k$ space and then Fourier transform into $x$ space.

$$\tilde{n}_1(\vec{k},t) = \omega_p^2 \int_0^t [\tilde{n}_1(\vec{k},t_1) - Z_i(t_1 - t)g(\vec{k}(t_1 - t))] dt_1$$

$$g(\vec{u}) = \int f_0(\vec{v})e^{i\vec{u} \cdot (\vec{v} + \vec{v}_0)} d^3v$$
Debugging Codes With Formula \( v_{i,z} = 5\sigma_s \) \( \psi = \pi \)

Numerical and Analytical Solution for Lorenzian
Comparison of Gauss.&Loren for $v_{i,z} = 0 \quad \psi = \pi$

Comparison of Gauss. & Loren for  $\nu_{i,z} = 0.5\sigma_s \quad \psi = \pi$


$ne^*RD^3$  

$R_{nex}(x, 1.572)$  

$nezg_1,3$  

$- 6.699 \times 10^{-5}$  

$z/RD$
Comparison of Gauss.&Loren for $\nu_{i,z} = \sigma_s \quad \psi = \pi$

Comparison of Gauss. & Loren for $v_{i,z} = 5\sigma_s \quad \psi = \pi$
2D Mapping of Electron Density

Lorenzian $v_z=5\sigma$, $x=y=0.41rD$
Summary

• The analytical result reproduces the Debye screening formula for an rest ion and for slow ions ($v_{0i} < \sigma_i$) it is close to the numerical Gaussian result. For fast ions, the Lorenzian approximation gives much smoother result than Gaussian distribution.

• The wavelength dependence of the electrons’ response oscillates about one plasma oscillation period. After that, oscillation only remains for small wavelength due to Landau Damping.

• The numerical Gaussian solution might be relatively fast compared with the PIC simulation and can be used for a quick estimation.