Effects on Hadron Beam from Electron Bunches
(Time dependent focusing effects)

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Outline

- Introduction
- Description of the formalism: betatron motion in the presence of small focusing variations.
- Emittance growth due to slipping electron bunch.
- Emittance growth due to random electron bunch by bunch variations:
  - Peak current variations;
  - Arriving time variations.
- Summary

Introduction

- In the electron cooling section, ions get focused by the electrons. The focusing strength from electrons can vary from turn to turn due to bunch charge jitters, arriving time jitters and electron bunch slipping with respect to ion bunch.
- The time-varying focusing can change an ion's Courant-Snyder invariant (action) as calculated from the time-independent linear lattice, which in combination of amplitude detuning, can lead to emittance growth of the ion beam.

Equations of Motion

$$H(x,p,\theta)=\frac{1}{2}\left(p^2+Q^2x^2\right)+\frac{1}{2}\delta_p(\theta)Qx^2\varepsilon(\theta)$$

$$\bar{H}(x,p,\theta)=\frac{1}{2\pi}\int_{\theta-\pi}^{\theta+\pi}H(x,p,\theta_1)d\theta_1\approx\frac{1}{2}\left[p^2+\tilde{Q}^2x^2\right]+\frac{1}{4\pi}\tilde{Q}x^2\delta\varepsilon(\theta)$$

$$x=\sqrt{\frac{2J}{\tilde{Q}}}\sin(\psi)\qquad p=\sqrt{2J\tilde{Q}}\cos(\psi)\qquad \tilde{Q}\equiv Q+\frac{\langle\varepsilon\rangle}{4\pi}$$

$$\bar{H}(\psi,J,\theta)=\tilde{Q}J+\frac{\delta\varepsilon(\theta)}{2\pi}J\sin^2\psi\qquad \langle\varepsilon\rangle=\frac{2\pi}{\theta}\int_0^{\theta}\delta_p(\theta_1)\varepsilon(\theta_1)d\theta_1$$

$$\boxed{\begin{aligned}\frac{dJ}{d\theta}&=-\frac{\partial\bar{H}}{\partial\psi}=-\frac{J}{2\pi}\sin(2\psi)\delta\varepsilon(\theta)\\ \frac{d\psi}{d\theta}&=\frac{\partial\bar{H}}{\partial J}=\tilde{Q}+\frac{1}{2\pi}\delta\varepsilon(\theta)\sin^2\psi\end{aligned}}\qquad \delta\varepsilon(\theta)=\varepsilon(\theta)-\langle\varepsilon\rangle$$

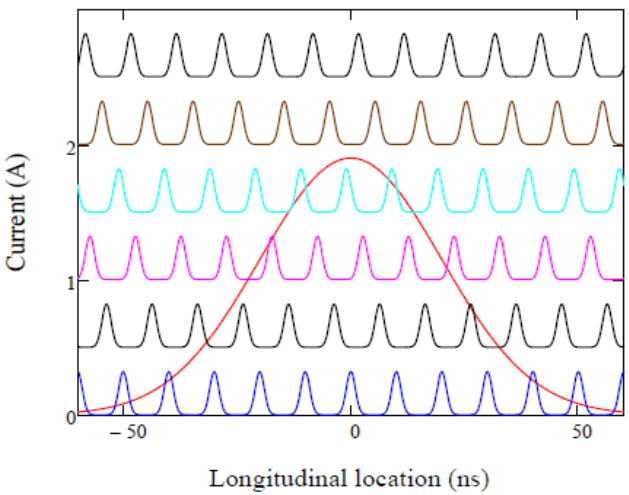
Leading Order Solution

$$J\left(N,\hat{\psi}\right)=J\left(0\right)\exp\left[u\left(N,\hat{\psi}\right)\right]$$

$$\psi_N \approx 2\pi\tilde{Q}N+\hat{\psi}-\frac{1}{2}\sum_{i=0}^{N}\cos\left[4\pi\tilde{Q}i+2\hat{\psi}\right]\delta\varepsilon_i$$

$$u\left(N,\hat{\psi}\right)\approx \underbrace{-\sum_{k=0}^{N}\sin\left(4\pi k\tilde{Q}+2\hat{\psi}\right)\delta\varepsilon_k}_{u_1}+\underbrace{\frac{1}{2}\left[\sum_{k=0}^{N}\cos\left(4\pi k\tilde{Q}+2\hat{\psi}\right)\delta\varepsilon_k\right]^2}_{u_2}$$

Focusing Strength from Slipping Electron Beam



$$I_e(t) = \frac{Q_e}{\sqrt{2\pi}\sigma_e} \sum_{n=-\infty}^{\infty} \exp\left[-\frac{(t - n\cdot\Delta T_e)^2}{2\sigma_e^2}\right]$$

$$\alpha = \frac{2r_0\beta_{avg}}{\gamma^3\beta^2 a_e^2}\frac{L_{cool}}{\beta c e}$$

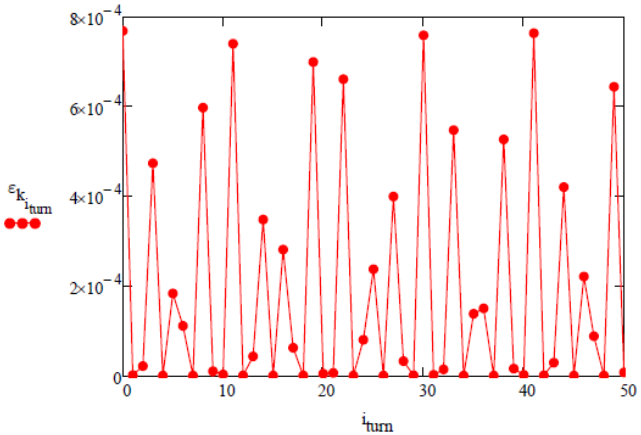
$$\varepsilon_M = \alpha \cdot I_e(M,0)$$

$$\approx 4\pi\cdot\Delta\nu_{peak}\cdot\exp\left\{-\frac{\left[M\cdot\nu_e - Round(M\cdot\nu_e)\right]^2\Delta T_e^2}{2\sigma_e^2}\right\} \quad \text{for } \Delta T_e >> \sigma_e$$

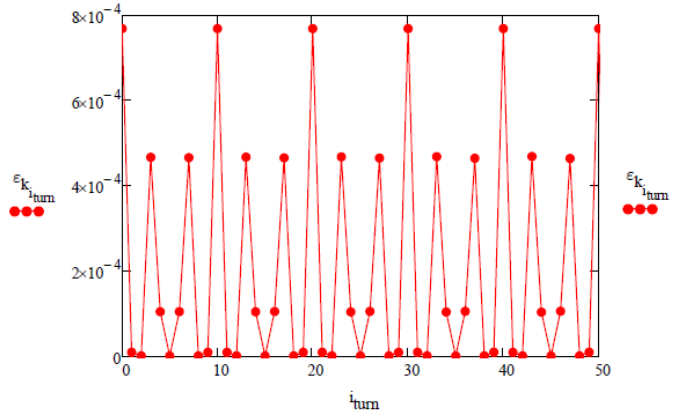
$$\nu_e = \mathrm{mod}\left(\frac{f_{e,rep}}{f_{i,rev}}, 1\right)$$

$$\Delta\nu_{peak} = \frac{N_e}{2\pi\sqrt{2\pi}\sigma_e}\frac{r_0\beta_{avg}}{\gamma^3\beta^3 a_e^2}\frac{L_{cool}}{c}$$

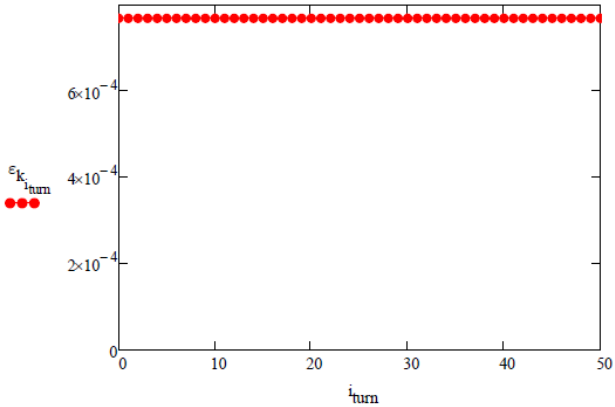
$\nu_e = 0.36613$



$\nu_e = 0.3$



$\nu_e = 0$



Fourier Decomposition

$$\varepsilon(t) = \alpha \cdot I_e(t)$$

$$I_e(t) = \frac{Q_e}{\sqrt{2\pi}\sigma_e} \sum_{n=-\infty}^{\infty} \exp\left[-\frac{(t - n\cdot\Delta T_e)^2}{2\sigma_e^2}\right] = \frac{2Q_e}{\Delta T_e}\left\{\frac{1}{2} + \sum_{k=1}^{\infty} \exp\left(-\frac{2k^2\pi^2\sigma_e^2}{\Delta T_e^2}\right)\cos\left(\frac{2k\pi t}{\Delta T_e}\right)\right\}$$

$$\langle I_e \rangle_{avg} = \frac{2Q_e}{\Delta T_e}\left\{\frac{1}{2} + \sum_{k=1}^{\infty} \Theta(N,\nu_e)\exp\left(-\frac{2k^2\pi^2\sigma_e^2}{\Delta T_e^2}\right)\right\}$$

$$\Theta(N,\nu_e) \equiv \frac{1}{2(N+1)}\left\{1 + \frac{\sin\left[(2N+1)k\pi\nu_e\right]}{\sin(k\pi\nu_e)}\right\}$$

$$\boxed{\delta\varepsilon_l = \alpha \cdot \left[I_e(lT_{rev}) - \langle I_e \rangle_{avg}\right] = \frac{2\alpha Q_e}{\Delta T_e}\sum_{k=1}^{\infty} \exp\left(-\frac{2k^2\pi^2\sigma_e^2}{\Delta T_e^2}\right)\cdot\left[\cos\left(\frac{2k\pi l T_{rev}}{\Delta T_e}\right) - \Theta(N,\nu_e)\right]}$$

Fourier Decomposition Continue

$$\nu_e \neq 0 \qquad g_k \equiv (2\pi)^{\frac{3}{2}} \frac{\sigma_e \Delta\nu_{peak}}{\Delta T_e} \mathrm{e}^{-\frac{2k^2\pi^2\sigma_e^2}{\Delta T_e^2}}$$

$$u_1(N,\hat{\psi}) = -2\sum_{k=1}^{\infty} g_k \left\{ \sum_{l=0}^{N} \sin\left[2\pi l \left(2\tilde{Q} + \frac{kT_{rev}}{\Delta T_e} \right) + 2\hat{\psi} \right] + \sum_{l=0}^{N} \sin\left[2\pi l \left(2\tilde{Q} - \frac{kT_{rev}}{\Delta T_e} \right) + 2\hat{\psi} \right] \right\}$$

$$u_2(N,\hat{\psi}) \approx \sum_{k=1}^{\infty} g_k^2 \sum_{m,n=0}^{N} \left\{ \cos\left[2\pi(n+m)\cdot\left(2\tilde{Q} + \frac{T_{rev}}{\Delta T_e} k \right) + 4\hat{\psi} \right] + \cos\left[2\pi(n+m)\cdot\left(2\tilde{Q} - \frac{T_{rev}}{\Delta T_e} k \right) + 4\hat{\psi} \right] \right.$$
$$\left. + \cos\left[2\pi(n-m)\cdot\left(2\tilde{Q} + \frac{T_{rev}}{\Delta T_e} k \right) \right] + \cos\left[2\pi(n-m)\cdot\left(2\tilde{Q} - \frac{T_{rev}}{\Delta T_e} k \right) \right] \right\}$$

Resonant condition: $\left(2\tilde{Q}\right)_{frac.} \pm k\nu_e = K$

Resonant Driving from Slipping Electron Bunch

Integer, resonance order

Integer

$$\left(2\tilde{Q}\right)_{frac.} \pm k\nu_e = K$$

$$\nu_e = \mathrm{mod}\left(\frac{f_{e,rep}}{f_{i,rev}}, 1\right)$$

$$J_{avg}\left(N\right) = J\left(0\right)\frac{1}{2\pi}\int_0^{2\pi}\exp\left\{-2g_k N\sin\left(2\hat{\psi}\right) + g_k{}^2 N^2\left[1+\cos\left(4\hat{\psi}\right)\right]\right\}d\hat{\psi}$$

$$\approx \frac{J\left(0\right)}{2\pi}\int_0^{2\pi}\left\{1 + g_k{}^2 N^2 + 2g_k{}^2 N^2\sin^2\left(2\hat{\psi}\right)\right\}d\hat{\psi}$$

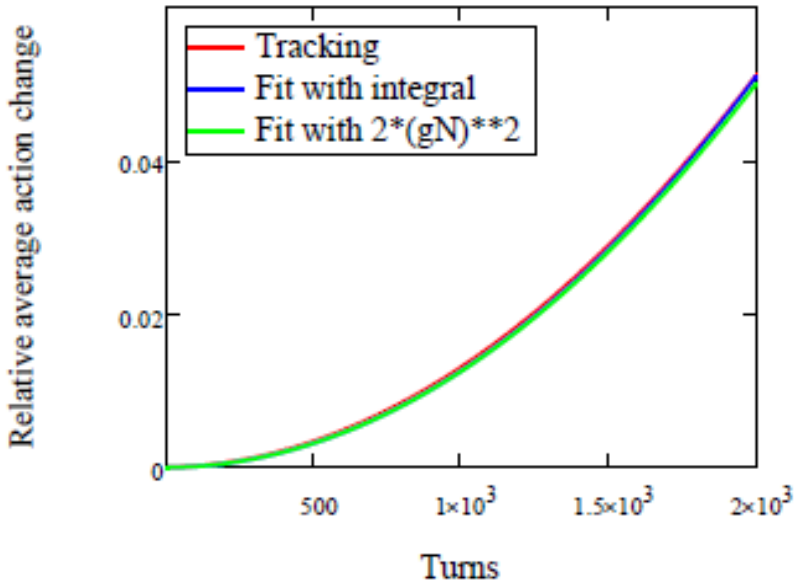
$$= J\left(0\right)\left(1 + 2g_k{}^2 N^2\right)$$

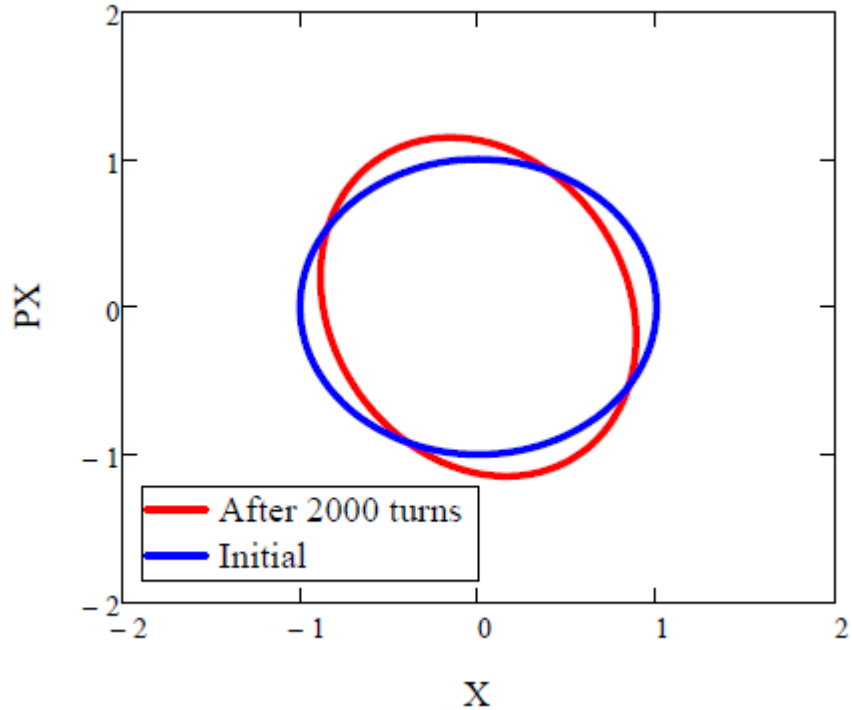
$$g_k \equiv \left(2\pi\right)^{\frac{3}{2}}\frac{\sigma_e \Delta\nu_{peak}}{\Delta T_e}\mathrm{e}^{-\frac{2k^2\pi^2\sigma_e^2}{\Delta T_e^2}}$$

1D Tracking

$$P = \frac{p}{\sqrt{Q}} = \sqrt{2J}\cos\psi \qquad X = x\sqrt{Q} = \sqrt{2J}\sin\psi$$

$$\begin{cases} X_n(k+1) = \cos(2\pi Q_x) X_n(k) + \sin(2\pi Q_x) P_n(k) \\ P_n(k+1) = \cos(2\pi Q_x) P_n(k) - \sin(2\pi Q_x) X_n(k) + \varepsilon_k X_n(k+1) \end{cases}$$





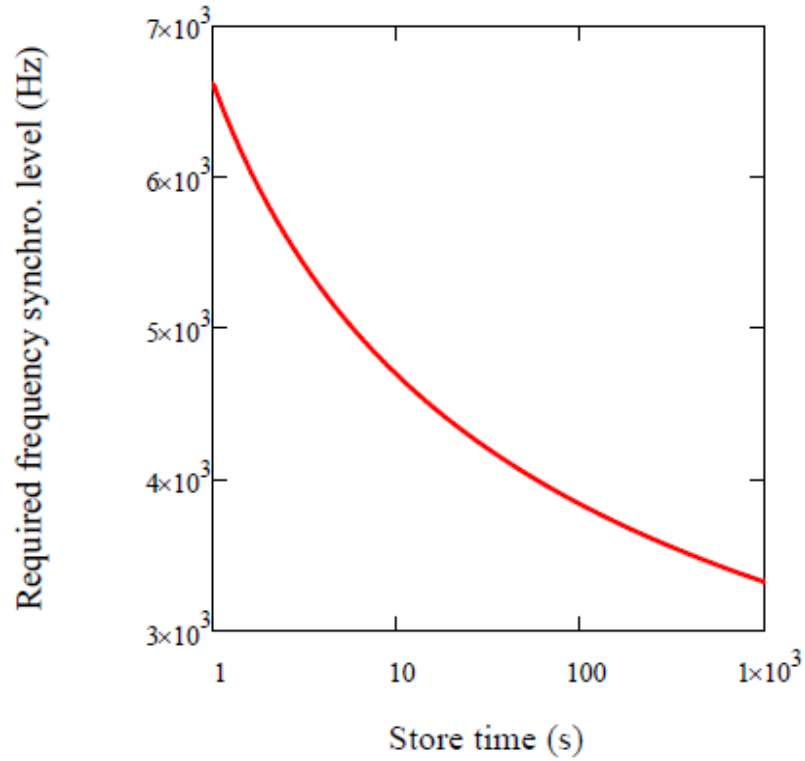
Tolerance on ν_e

- For phase locked scenario, ν_e will be exactly zero.
- But in principal, small deviation from zero, which only leads to high order resonances, should be tolerable.

$$g_k = (2\pi)^{\frac{3}{2}} \frac{\sigma_e \Delta\nu_{peak}}{\Delta T_e} \mathrm{e}^{-\frac{2k^2\pi^2\sigma_e^2}{\Delta T_e^2}} < \frac{1}{\sqrt{2}N_{store}}$$

$$k > \frac{\Delta T_e}{\sqrt{2}\pi\sigma_e} \sqrt{\ln\left[\frac{\sqrt{2}(2\pi)^{\frac{3}{2}} N_{store}\sigma_e \Delta\nu_{peak}}{\Delta T_e}\right]}$$

$$\delta f_{rep} < \sqrt{2}\pi \cdot \frac{\sigma_e}{\Delta T_e} \cdot f_{rev} \cdot \min\left(2\tilde{Q}_{dec}, 1-2\tilde{Q}_{dec}\right) \times \left\{\ln\left[\frac{\sqrt{2}(2\pi)^{\frac{3}{2}} \sigma_e \Delta\nu_{peak} N_{store}}{\Delta T_e}\right]\right\}^{-\frac{1}{2}}$$



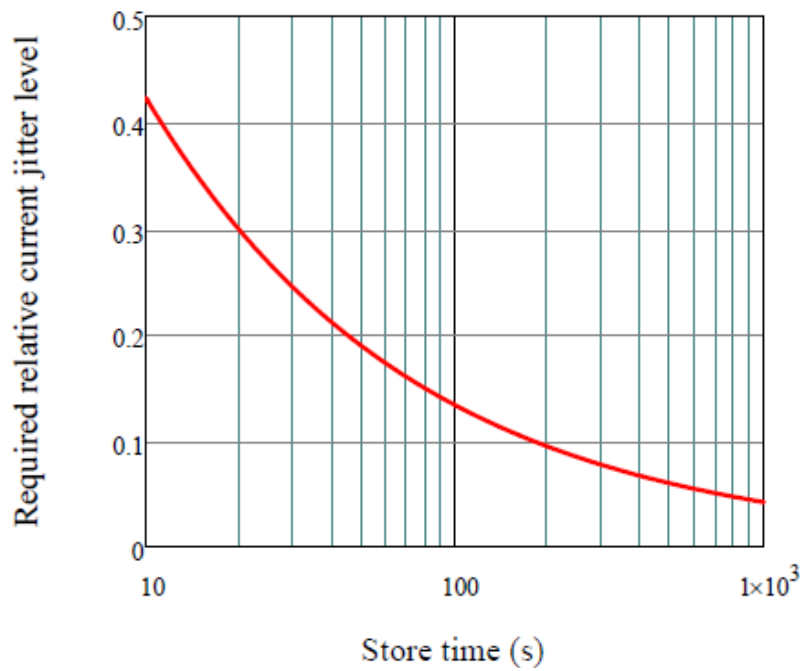
Emittance Growth due to Peak Current Jitter
(Reference: C-A/AP/#363)

For non-correlated jitters:

$$J_N = J_0 \exp\left[N \frac{\langle \delta\varepsilon^2 \rangle}{2} \right] \qquad \langle \delta\varepsilon^2 \rangle = 16\pi^2 \langle \Delta\nu_{peak} \rangle^2 \frac{\langle \delta I_e(0)^2 \rangle}{\langle I_e(0) \rangle^2}$$

$$\tau_{grow} = \frac{T_{rev}}{8\pi^2 \langle \Delta\nu_{peak} \rangle^2 \cdot \frac{\langle \delta I_e(0)^2 \rangle}{\langle I_e(0) \rangle^2}} \Rightarrow \frac{\sqrt{\langle \delta I_e(0)^2 \rangle}}{\langle I_e(0) \rangle} < \frac{1}{2\pi \langle \Delta\nu_{peak} \rangle} \sqrt{\frac{T_{rev}}{2t_{store}}}$$

- To keep the maximal betatron oscillation action growth within one e fold for 1000 seconds, the rms peak current jitter is required to be under 4% in absence of the synchrotron oscillation.
- The growth rate reduces by a factor of three after averaging over the synchrotron oscillation period and hence the requirement is relaxed to 7%.

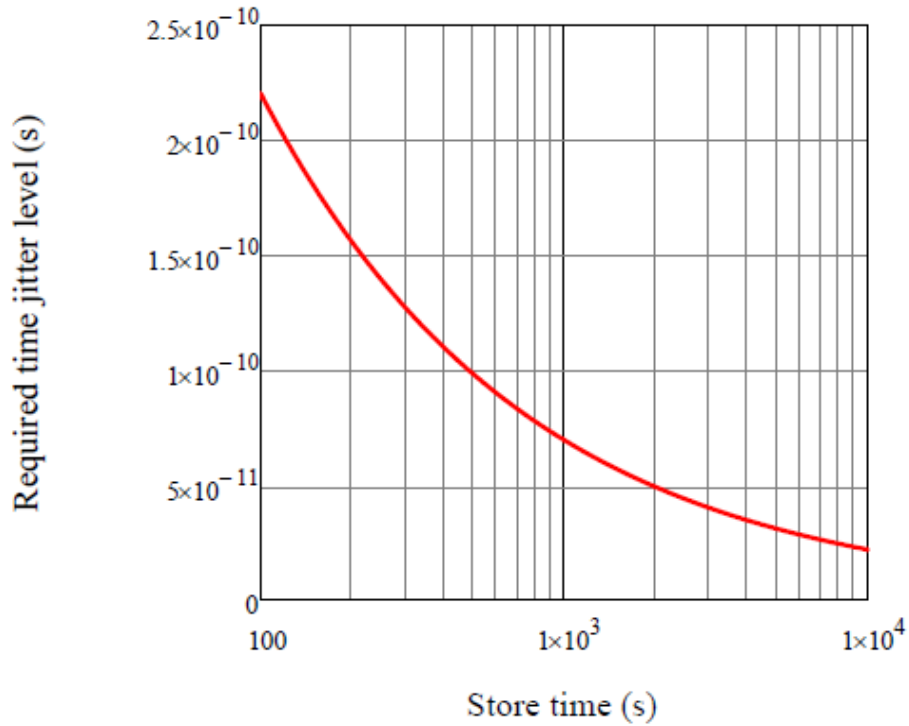


Emittance Growth due to Arriving Time Jitter

$$\delta\varepsilon(\tau+\delta_t)=\alpha\cdot\left[I_e(\tau+\delta_t)-I_e(\tau)\right]=\alpha\cdot\frac{d}{d\tau}I_e(\tau)\cdot\delta_t \qquad \alpha=\frac{2r_0\beta_{avg}}{\gamma^3\beta^2a_e^2}\frac{L_{cool}}{\beta ce}$$

$$\tau_{grow}=\frac{e}{8\pi^2}T_{rev}\frac{\sigma_e^2}{\langle\delta_t^2\rangle}\frac{1}{\Delta\nu_{peak}^2}\Rightarrow\sqrt{\langle\delta_t^2\rangle}<\frac{1}{2\pi}\frac{\sigma_e}{\Delta\nu_{peak}}\sqrt{\frac{eT_{rev}}{2t_{store}}}$$

- To keep the maximal betatron oscillation action growth within one e fold for 1000 seconds, the rms arriving time jitter is required to be under 70 ps in absence of the synchrotron oscillation.
- The growth rate reduces by a factor of three after averaging over the synchrotron oscillation period and hence the requirement is relaxed to 120 ps.



Summary

- Slipping electron bunch can be considered as a bunch of rf quadruples, which can drive ion beam resonantly and lead to its emittance growth if the resonant condition, $\left(2\tilde{Q}\right)_{frac.} \pm k\nu_e = K$, is satisfied. We plan to avoid the resonant driving by choosing $\nu_e = 0$
- Peak current jitters of the electron bunches lead to random focusing variations to the ions in the cooling section, which can cause ion beam emittance growth. The rms current variation should be under 7% to keep the ion emittance growth time longer than 1000 seconds.
- Arriving time jitters of the electron bunch will also cause focusing variations seen by ions and hence ion beam emittance growth. The rms arriving time variation should be under 120 ps to keep the ion emittance growth time longer than 1000 seconds.

Thank you

Averaging over Synchrotron Amplitude

$$\frac{1}{\tau_{grow}(A,t)} = \eta n_e^2(z) = \eta n_e^2\left(A\cos(\omega_s t)\right)$$

$$\Lambda_{red} = \tau_0 \left\langle \frac{1}{\tau_{grow}} \right\rangle_{T_s,A}$$

$$\frac{1}{\tau_0} = \frac{1}{\tau_{grow}(A=0)} = \eta n_e^2(0)$$

$$\left\langle \frac{1}{\tau_{grow}} \right\rangle_{T_s,A} = \frac{1}{T_s A_m} \int_0^{A_m} dA \int_0^{T_s} \frac{1}{\tau_{grow}(A,t)} dt$$

$$f_i(A,\psi_s) = \frac{1}{2\pi A_m A} H(A_m - A)$$

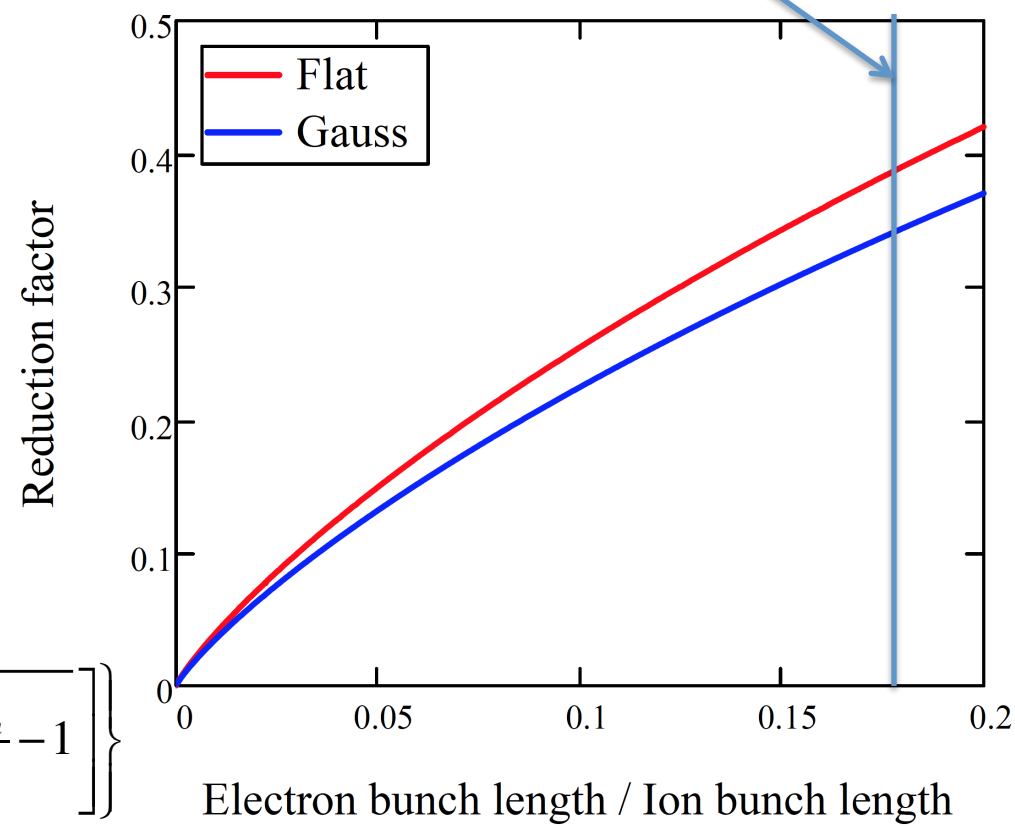
where LEReC is

Gassian distributed electron bunch sitting at ion bunch center:

$$\Lambda_{red} = \frac{\sigma_e}{2\sqrt{\pi} A_m} \int_0^1 \frac{Erf\left(\frac{A_m}{\sigma_e} x\right) + Erf\left(\frac{A_m}{\sigma_e} x\right)}{x\sqrt{1-x^2}} dx$$

Evenly (flat) distributed electron bunch sitting at ion bunch center:

$$\Lambda_{red} = \frac{2}{\pi} \frac{l_e}{A_m} \left\{ \frac{A_m}{l_e} \arcsin\left(\frac{l_e}{A_m}\right) + \ln\left[\frac{A_m}{l_e} + \sqrt{\frac{A_m^2}{l_e^2} - 1}\right] \right\}$$



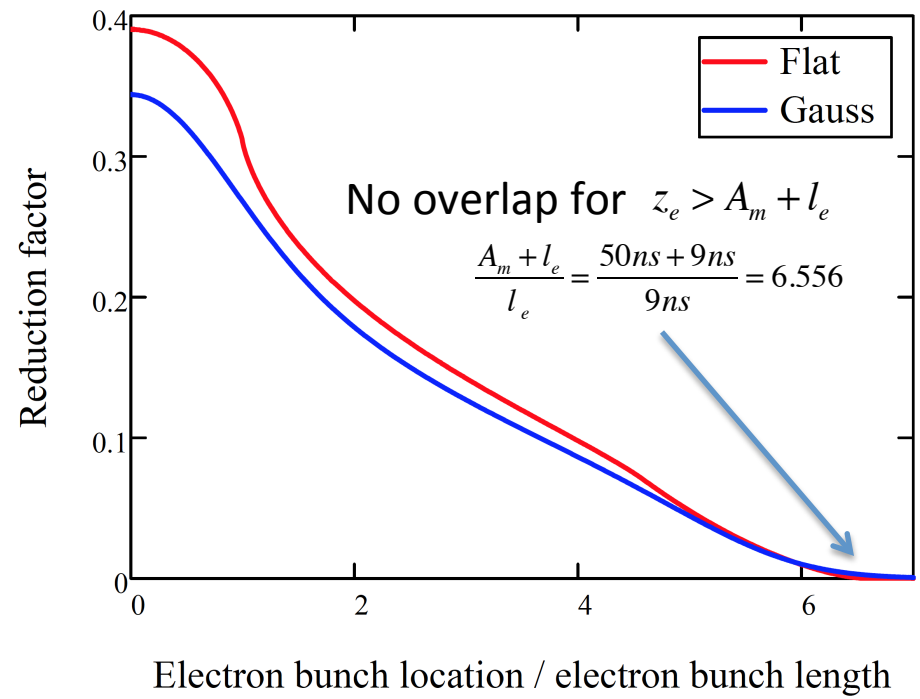
Averaging over Synchrotron Amplitude continue

Evenly (flat) distributed electron bunch sitting at arbitrary location:

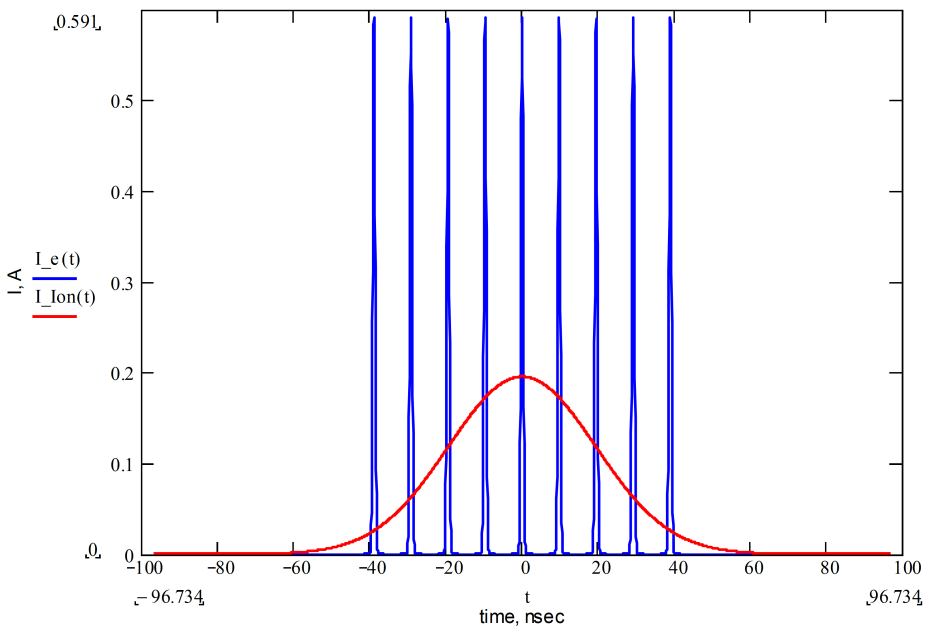
$$\Lambda_{red} = \frac{1}{\pi}\int_0^1 dx \frac{1}{x\sqrt{1-x^2}}\left\{\left(x-\frac{z_e-l_e}{A_m}\right)H\left(x-\frac{z_e-l_e}{A_m}\right)-\left(x-\frac{z_e+l_e}{A_m}\right)H\left(x-\frac{z_e+l_e}{A_m}\right)\right.$$
$$\left.+\left(x+\frac{z_e-l_e}{A_m}\right)H\left(-x-\frac{z_e-l_e}{A_m}\right)-\left(x+\frac{z_e+l_e}{A_m}\right)H\left(-x-\frac{z_e+l_e}{A_m}\right)\right\}$$

Gaussian distributed electron bunch sitting at arbitrary location:

$$\Lambda_{red} = \frac{\sigma_e}{2\sqrt{\pi}A_m}\int_0^1 \frac{Erf\left(\frac{A_m}{\sigma_e}x-\frac{z_e}{\sigma_e}\right)+Erf\left(\frac{A_m}{\sigma_e}x+\frac{z_e}{\sigma_e}\right)}{x\sqrt{1-x^2}}dx$$



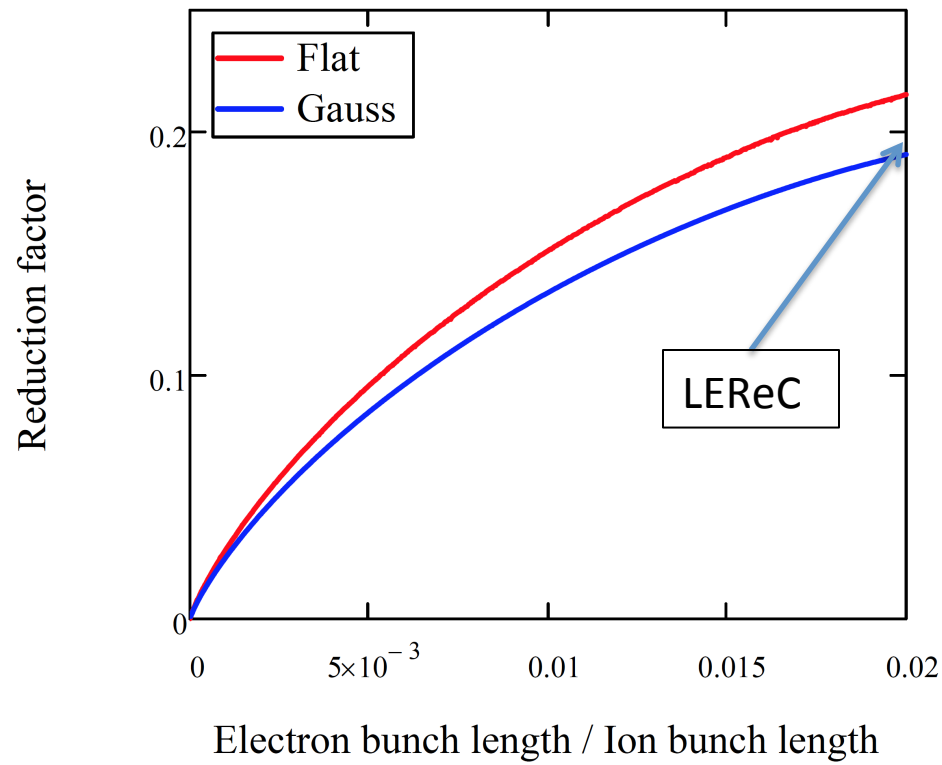
Averaging Synchrotron Amplitude Continue 2



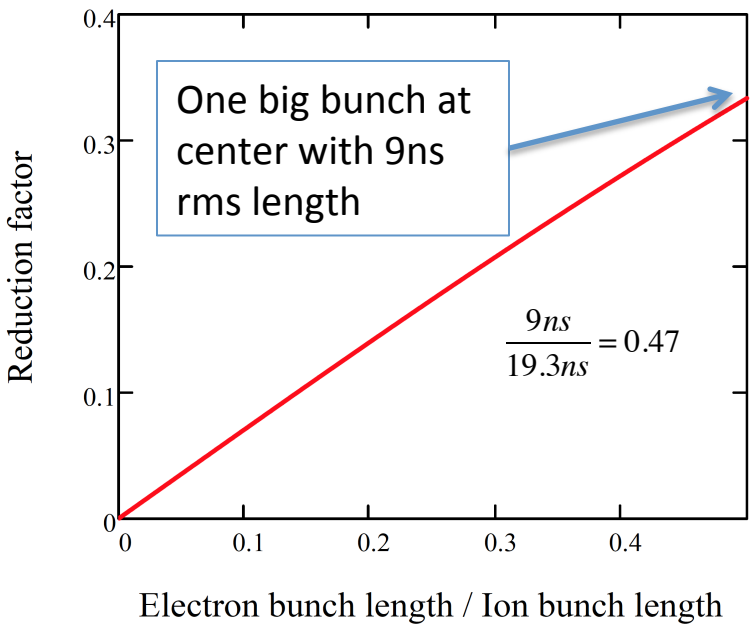
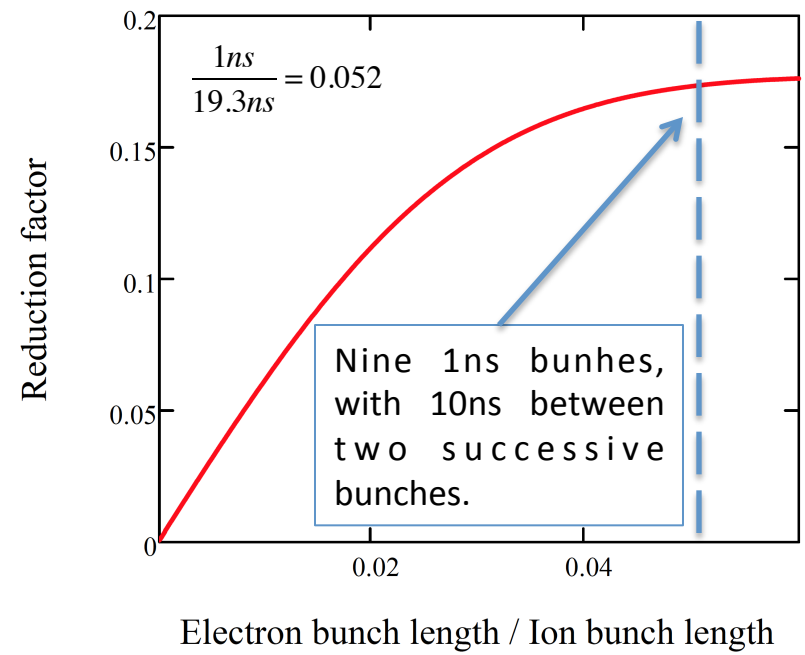
- Instead of a factor of 3 reduction as predicted by putting all the electron bunch to the center, this more realistic calculation predicts the reduction factor is ~5.

For well-separated electron bunches:

$$\left\langle \frac{1}{\tau_{total}} \right\rangle_{T_s, A} \approx \sum_{i=1}^{N_{bunch}} \left\langle \frac{1}{\tau_i(z_e)} \right\rangle_{T_s, A}$$



Gaussian Ion Distribution

$$f_i(A,\psi_s) = \frac{1}{2\pi\sigma_A^2}\exp\left(-\frac{A^2}{2\sigma_A^2}\right) \qquad g_3(x) = \frac{4\bar{z}_e x}{\sqrt{2x^2+\bar{\sigma}_e^2}} \qquad \bar{z}_e \equiv \frac{z_e}{\sigma_e} \qquad \bar{\sigma}_e \equiv \frac{\sigma_e}{\sigma_A}$$

$$\Lambda_{red} = \frac{2}{\pi}e^{-\bar{z}_e^2}\int_0^1 \frac{1}{\sqrt{1-x^2}}\cdot\frac{1}{1+\frac{2x^2}{\bar{\sigma}_e^2}}\left[1+\sqrt{\frac{\pi}{8}}\cdot g_3(x)\exp\left(\frac{g_3(x)^2}{8}\right)erf\left(\frac{g_3(x)}{2\sqrt{2}}\right)\right]dx$$