Exact Solutions for the n-axis and Spin Tune for Model Storage Rings with Siberian Snakes

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- New <u>nonperturbative</u> formalism MILES

  based exclusively on field-theoretic transformation of vector field

- List of Models (exact solutions)
  - SRM
  - SRM with Partial Type 3 Snake
  - Vertical resonance driving term and 1 Snake
  - Vertical resonance driving term and 2 Snakes
  - SRM with 1 Snake
  - SRM with 2, 4, 6 ... Snakes

- Spin Tune
  - Yokoya (1988) conjecture $\nu = 1/2$
    - confirm
  - Snake resonances (Lee & Tepikian 1986)
    - confirm
  - *exceptional orbits — new!*

- **Miles**

$$\boldsymbol{\sigma}\cdot\boldsymbol{n}(z_f) = M\,\boldsymbol{\sigma}\cdot\boldsymbol{n}(z_i)\,M^{-1}$$

$$M = \begin{pmatrix} f & -g^* \\ g & f^* \end{pmatrix}$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{n} = \begin{pmatrix} n_3 & n_- \\ n_+ & -n_3 \end{pmatrix}$$

$$\begin{aligned} n_3(\boldsymbol{\phi}_* + \boldsymbol{\mu}) &= (ff^* - gg^*)n_3(\boldsymbol{\phi}_*) - f^*g^*n_+(\boldsymbol{\phi}_*) - fgn_-(\boldsymbol{\phi}_*) \\ n_+(\boldsymbol{\phi}_* + \boldsymbol{\mu}) &= 2f^*gn_3(\boldsymbol{\phi}_*) + f^{*2}n_+(\boldsymbol{\phi}_*) - g^2n_-(\boldsymbol{\phi}_*) \end{aligned}$$

- Fourier Series

$$\begin{aligned} n_3(\boldsymbol{\phi}_*) &= \sum_{\boldsymbol{m}} a_{\boldsymbol{m}}\, e^{i\boldsymbol{m}\cdot\boldsymbol{\phi}_*} \\ n_+(\boldsymbol{\phi}_*) &= \sum_{\boldsymbol{m}} b_{\boldsymbol{m}}\, e^{i\boldsymbol{m}\cdot\boldsymbol{\phi}_*} \end{aligned}$$

- Solution at other azimuths obtained by tracking

- SRM with 1 Snake

$$\boldsymbol{W}_{\rm arc} = \nu_0 \boldsymbol{e}_3 + \epsilon\,(\cos\phi\,\boldsymbol{e}_1 + \sin\phi\,\boldsymbol{e}_2) \qquad \sim \text{SRM}$$

$$\boldsymbol{W}_{\rm Snake} = \pi\delta_p(\theta)\,(\cos\xi\,\boldsymbol{e}_1 + \sin\xi\,\boldsymbol{e}_2) \qquad \sim \text{ angle } \xi$$

$$\Omega \equiv \sqrt{(\nu_0 - Q)^2 + \epsilon^2}$$

$$\eta \equiv \frac{\epsilon}{\Omega}\sin\pi\Omega \qquad\qquad -1 \le \eta \le 1$$

$$M = \begin{pmatrix} -\eta\, e^{-i(\phi_* - \xi + \mu/2)} & -i\sqrt{1-\eta^2}\, e^{-i(\kappa + \xi + \mu/2)} \\ -i\sqrt{1-\eta^2}\, e^{i(\kappa+\xi+\mu/2)} & -\eta\, e^{i(\phi_* - \xi + \mu/2)} \end{pmatrix}$$

$$n_3 = (1-\eta^2)\,a$$

$$n_+ = i g\, b$$

$$a = 2\sum_{m=\text{odd}} a_m \sin[m(\phi - \xi)]$$

$$b = b_0 + 2\sum_{m=\text{even}} b_m \cos[m(\phi-\xi)]$$

"Sine-Factorial"

$$\delta \equiv Q - \frac{1}{2}$$

$$S_m(\delta) = \sin(\pi\delta)\sin(2\pi\delta)\cdots\sin(m\pi\delta)$$

$$C_m(\delta) = \cos(\pi\delta)\cos(2\pi\delta)\cdots\cos(m\pi\delta)$$

Solution

$$a_m = \frac{1}{\cos(m\pi\delta)} \sum_{k=0}^{\infty} \frac{C^2_{m/2+k}}{S_k S_{m+k}} (-1)^k \, \eta^{m+2k}$$

$$b_m = \sum_{k=0}^{\infty} \frac{C^2_{(m-1)/2+k}}{S_k S_{m+k}} (-1)^k \, (\eta \, e^{i\pi\delta})^{m+2k}$$

Bessel functions

$$J_m(\eta) = \sum_{k=0}^{\infty} \frac{1}{k!\,(m+k)!} (-1)^k \left(\frac{\eta}{2}\right)^{m+2k}$$

Asymptotic

$$\delta \lll 1 \qquad \Longrightarrow \qquad a_m \sim b_m \sim J_m\left(\frac{\eta}{\delta}\right)$$

- Spinor solution
- SODOM2 (Yokoya 1998)

$$\boldsymbol{n} = \Psi^\dagger \boldsymbol{\sigma} \Psi$$

$$\Psi = \begin{pmatrix} A \\ igB \end{pmatrix}$$

- Fourier Series

$$A = A_0 + 2 \sum_{m=\text{even}} A_m \cos[m(\phi_* - \xi)] + 2 \sum_{m=\text{odd}} A_m \sin[m(\phi_* - \xi)]$$

$$B = B_0 + 2 \sum_{m=\text{even}} B_m \cos[m(\phi_* - \xi)] - 2 \sum_{m=\text{odd}} B_m \sin[m(\phi_* - \xi)]$$

- "Sine-Bessel" functions

$$A_m = \sum_{k=0}^{\infty} \frac{e^{ik(m+k)\pi\delta}}{S_k S_{m+k}} (-1)^k \left(\frac{\eta}{2} e^{-i\pi\delta}\right)^{m+2k}$$

$$B_m = \sum_{k=0}^{\infty} \frac{e^{ik(m+k)\pi\delta}}{S_k S_{m+k}} (-1)^k \left(\frac{\eta}{2} e^{i\pi\delta}\right)^{m+2k}$$

- Spin Tune

$$\nu = \frac{1}{2} \qquad \Longleftarrow \qquad \underline{\textit{all}} \text{ nonresonant orbits} \quad |\eta| < 1$$

  - Yokoya conjecture (SSC 189, 1988)

- Snake resonances
  - small denominators in Fourier harmonics
    - 1 Snake

$$Q = \frac{\ell_0}{2\ell_1}$$

    - 2 Snakes

$$Q = \frac{2\ell_2 + 1}{2(2\ell_1 + 1)}$$

  - Lee & Tepikian ($\sim$ 1986)
  - **nonperturbative** proof
    - purely analytical
    - no tracking

- Exceptional Orbits
- $|\eta| = 1$

$$
\begin{aligned}
M &= \begin{pmatrix} e^{-i(\phi_* - \xi + \pi Q)} & \\ & e^{i(\phi_* - \xi + \pi Q)} \end{pmatrix} & \qquad \eta = -1 \\
\Psi &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \Longrightarrow \qquad \boldsymbol{n} = \text{vertical} \\
\nu &= \frac{\phi_* - \xi + \pi Q}{\pi}
\end{aligned}
$$

- spin tune depends <u>explicitly on orbit phase</u>
  - *signature of exceptional orbit*
  - *spin changes topology of orbit manifold*
    - ∗ normally neglected (Stern-Gerlach effect)
    - ∗ map algorithms <u>do not transform orbit</u>
      - · need <u>full spin-orbit CT</u>
      - · mistake in Yokoya calc (1986)
  - 2, 4, 6 ... Snakes, non-pointlike Snake

- comment from referee
  - Penning traps
  - CSGE (continuous Stern-Gerlach effect)
  - detection via shift of axial oscillation frequency

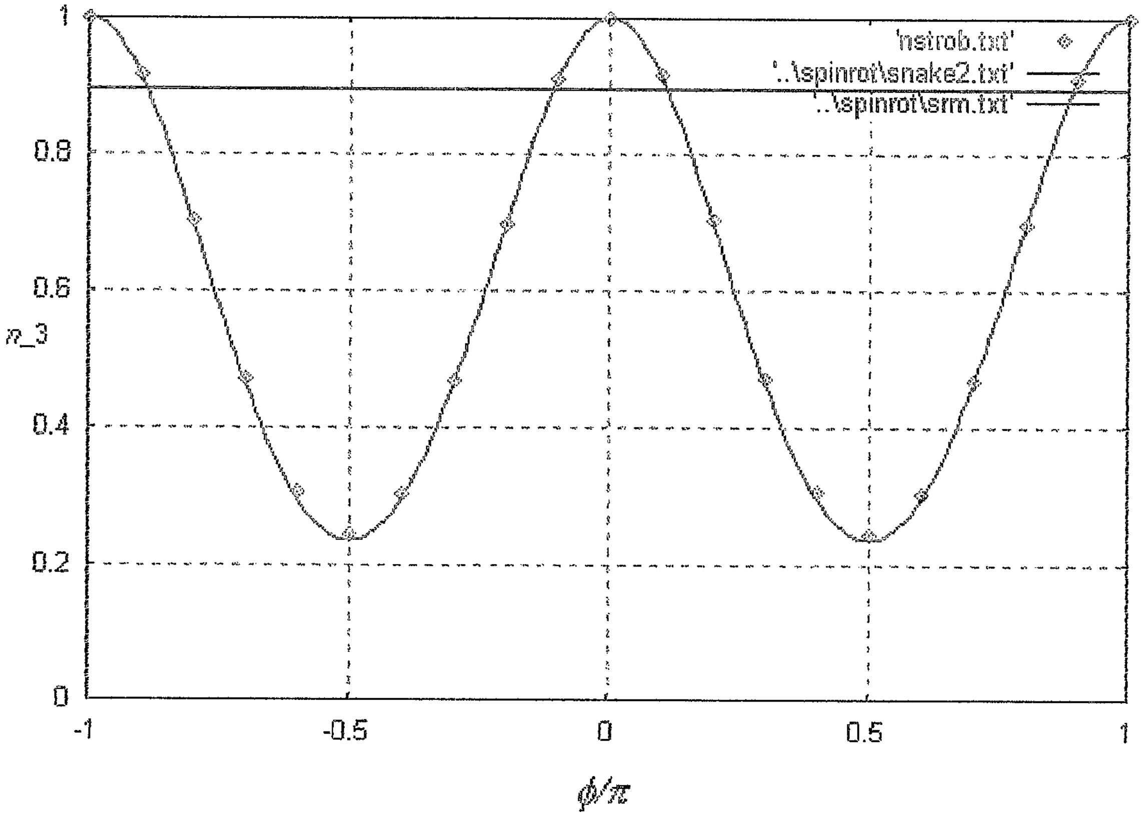
Conclusions

- New formalism MILES
  - nonperturbative
  - based exclusively on field-theoretic transformation of vector field

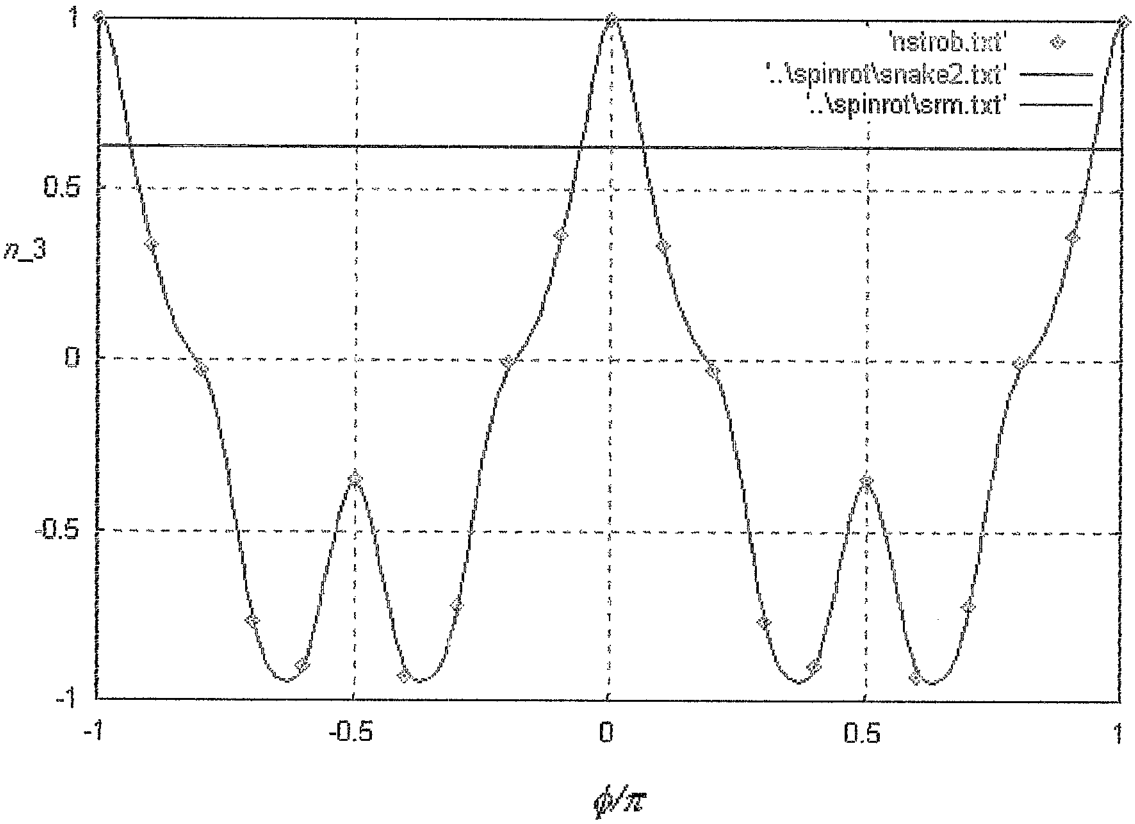
- exact solutions many models
  - Snakes
  - confirm conjectures
    * $\nu = 1/2$ (nonexceptional orbits)
    * Snake resonances

- *new mathematical functions*
  - sine-Bessel
  - sine-factorial

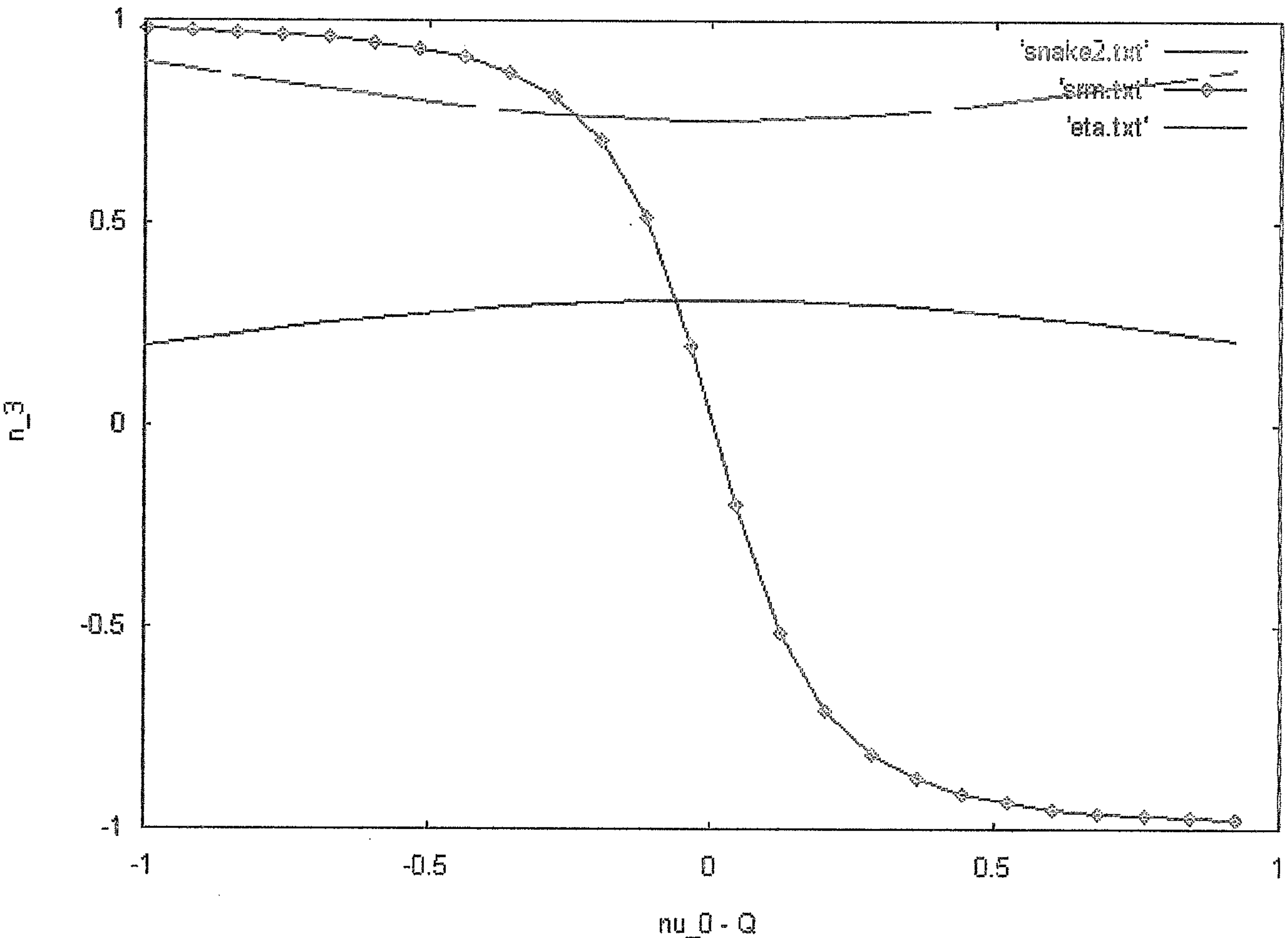
- phase-dependent spin tune
  - "exceptional orbits"
  - necessary for formal theory
  - rare
    * no reason for panic



eps = 0.2
delta = 0.1414...
nu_0 - Q = 0.4
eta = 0.289



eps = 0.5
delta = 0.1414...
nu_0 - Q = 0.4
eta = 0.66



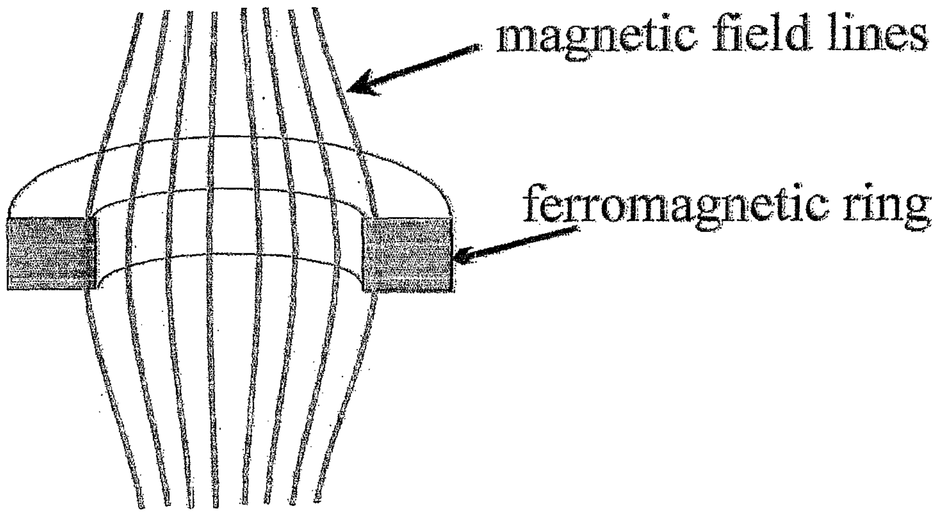
$\epsilon = 0.1$

$Q = 10.707\ldots$

$\delta = 0.207\ldots$

Detection of Spin Flips

In order to measure the Larmor-frequency we have to detect spin flips. This is accom with the help of a magnetic bottle (produced by a nickel ring). It couples the spin orie to the axial frequency of the ion (This is called "Continuous-Stern-Gerlach effect"):



The nickel ring intoduces a quadratic dependency of the magnetic field on the spatial coodinates (here the z-coordinate):

$$B = B_0 + B_2 z^2$$

Therefore the potential energy V is modified by the nickel ring as follows:

$$V(z) = V_{el} + V_{mag} = V_{el} + \mu(B_0 + B_2 z^2)$$

The additional square dependency of the potential energy on the z-coordinate thus le additional binding or anti-binding force (depending on the sign of the magnetic mom This makes it possible to observe spin-flips of the electron by measuring the axial fre