# Collective effects and Coulomb collisions in halo genesis and growth

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# Introduction

**Halo definition:** Lacking or controversial.

**Halo origin:** Initial distribution

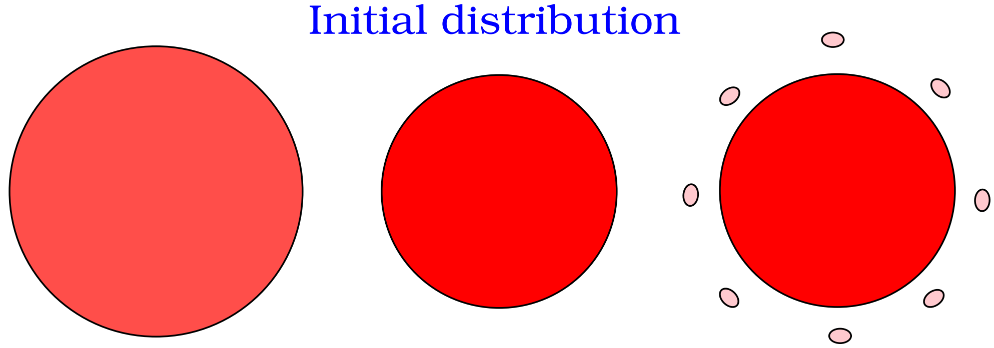
**Halo growth:** Collective motions and nonlinear resonances, collisional effects.

Halo test particles in KV beams stronlgy affected by mismatch oscillations.

**Linacs:** (ADS) short bunches 3D dynamics, few FODO cells ( $\sim 10^2$ ).

**Rings** (SIS-100) long bunches, 2D dynamics, $10^6 \sim 10^8$ visited FODO cells

Initial distribution



KV matched | Kv mismatched | Kv mismatched with satellites

**No halo** | **No halo** | **Halo**

# Halo dynamics

**ADS linac** We have developed 2D and 3D models for test particles dynamics.

**Core field:** analytical or PIC solution of Poisson-Vlasov based on FFT

**Analysis** of nonlinear resonances based of frequency map : tune footprints, resonace charts, invariant actions plots. Mismatch dominates.

**Diffusion:** resonance crossings due to mean field, collisional noise.

# Halo thermodynamics

**Mixing property** of full hamiltonian $H$ likely to hold.

**Relaxation** to Gibbs ensemble implies Maxwell Boltzmann for single particle distribution, equipartition of quadratic degrees of freedom.

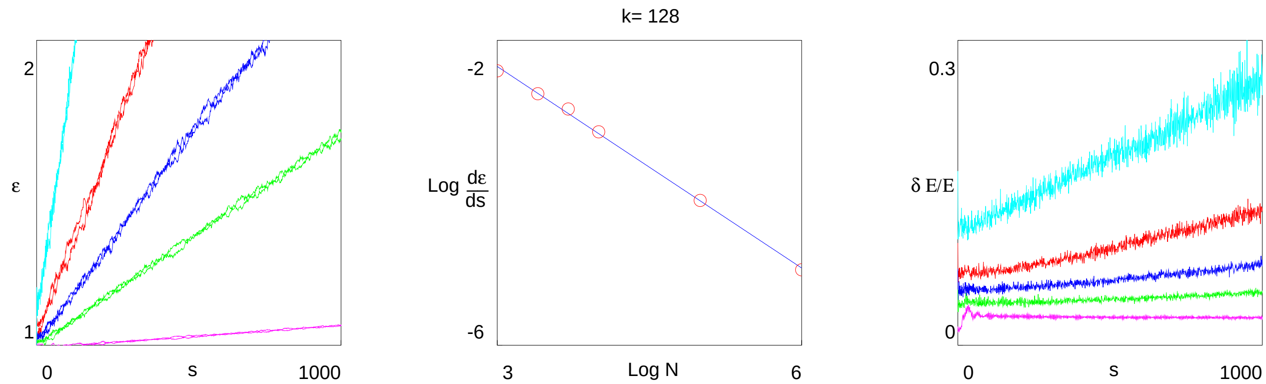
**Kinetic theory** Boltzmann or Landau assumptions need to be checked.

**Randomness** Fluctuations of physical (small angle collisions) or numerical (psedoparticles number in PIC) origin leads to diffusion.

**Numerical randomness** in PIC codes can be controlled.

**Physical randomness** due to collisions, is unavoidable and leads to the Maxwell-Boltzmann relaxation.

**Relaxation** We propose here a a molecular dynamics approach to investigate the relaxation, check scaling laws and limits of kinetic theory: Poisson-Vlasov-Fokker-Planck equation




**Noise error in PIC**. Left: linear emittance growth for a PIC solver. High perveance: $128 \times 128$ Fourier components and $2.5\,10^3, 5\,10^3, 10^, 10^5$ . Center: slope of emittance increase $d\epsilon/ds \propto N^{-1}$ Right: electric field error. Fraction of occupied cells $7\%$ .

# 1. The 2D model (coasting beam in costant focusing)

Longitudinal speed $v_0 \gg v_T$ transverse speed.

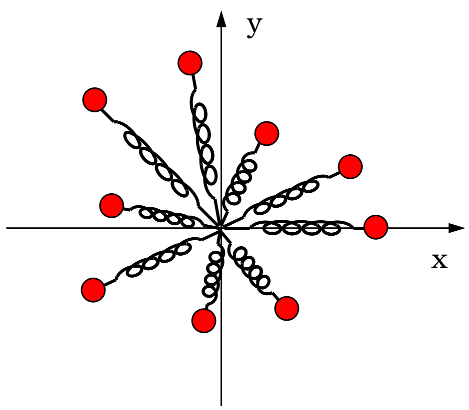
The 2D approximation assumes coherence along $z$ : particles organized in parallel wires: mass $m$ , charge $q$ per unit length.

**Ratios:** $q/m = e/m_p$ . Total charge per unit length $Q = N\,q = I/v_0$ fixed. Cell length $L$=1 m , core radius $R\sim 1$ mm .

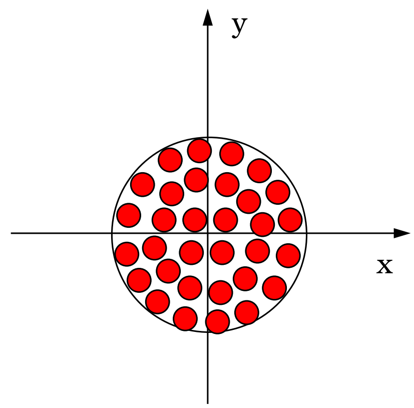
**Control parameters.** The model depends on two control parameters: the bare tune $\omega_0$ and the perveance $\xi$

$$\boxed{\omega_0^2 = \frac{k_0}{m v_0^2} = \frac{q}{m}\,\frac{B_1}{c v_0}}$$

$$\boxed{\xi = \frac{q}{m}\,\frac{2Q}{v_0^2}}$$




The forces




The equilibrium

**Equations of motion** For the 2D model

$$\frac{d^2 x_i}{ds^2} + \omega_0^2\, x_i = \frac{\xi}{N} \sum_{j \neq i}^{N} \frac{x_i - x_j}{r_{ij}^2} \qquad \frac{d^2 y_i}{ds^2} + \omega_0^2\, y_i = \frac{\xi}{N} \sum_{j \neq i}^{N} \frac{y_i - y_j}{r_{ij}^2}$$

where $N$ e number of pseudoparticles (wires).

**Scaling law** Letting $N_p$ be the number of protons per unit length. The density is $\rho_p = N_p/(\pi R^2)$ , the specific length is $\ell = \rho^{-1/3}$ The number $N_{\rm phys}$ of wires is equal to the number of particles in a cilinder of height $\ell$

$$N_{\rm phys} = N_p \ell = N_p^{2/3}\, (\pi R^2)^{1/3}$$

For $N_p = 10^8/\ \mathrm{mm}$ we obtain $\ell = 3{\times}10^{-3}\ \mathrm{mm}$ and $N_{\rm phys} \sim 3{\times}10^5$ .

**Relaxation times** From simulations depends linearly on $N$ . We prove this scaling holds within Landau' theory. Hence

$$\boxed{\tau_{\rm phys} = \frac{N_{\rm phys}}{N}\, \tau(N)}$$

**The Hamiltonian** for the 2D model reads

$$H = \sum_{i=1}^{N} H^{(i)}$$

where $H^{(i)}$ is the Hamiltonian for particle $i$

$$H^{(i)} = \frac{p_{x\,i}^2 + p_{y\,i}^2}{2} + \omega_0^2 \frac{x_i^2 + y_i^2}{2} + \frac{\xi}{2} V^{(i)}$$

**The electric potential** for particle $i$ splits into two parts (mean field and collisional)

$$V^{(i)} = -\frac{2}{N} \sum_{k, r_k > R_D} \log r_{i\,k} - \frac{2}{N} \sum_{k, r_k < R_D} \log r_{i\,k}$$

where $R_D$ is the Debye radius.

**Continuum limit** The collisional component vanishes in the limit $N \to \infty$ in the framework of Landau's theory.

**2D Debye shielding** A small perturbation to a uniform distribution is balanced sel-consistently by the thermal fluctuations. The potential $V$ of the perturbed field satisfies

$$\Delta V = -4\pi N\, e\,(\rho_S - \rho_{0\,S}) = -4\pi\rho_{0\,S}\left[\exp\left(-\frac{eV}{k_B T}\right) - 1\right] \simeq \frac{V}{\lambda_D^2}$$

The solution of the $2D$ Poisson equation is

$$\boxed{V(r) = 2Q\, K_0\left(\frac{r}{\lambda_D}\right)} \qquad \lambda_D^2 = \frac{k_B T}{4\pi N\, e^2 \rho_{0\,S}} = \frac{2\omega\epsilon}{\xi}\left(\frac{R}{4}\right)^2$$

**Asymptotic behaviour** The decay is exponential with the Debye length $\lambda_D$

$$V \simeq 2Q\frac{e^{-r/\lambda_D}}{\sqrt{r/\lambda_D}} \quad r \to \infty \qquad V \simeq -2Q\log\frac{r}{\lambda_D} \quad r \to 0$$

# 2. Mean field equations

Neglecting the collisional component the potential of the mean field is a solution of the Poisson equation

$$\boxed{\Delta V = -4\pi\rho_S} \qquad \rho_s(x,y) = \int \rho(x,p_x,y,p_y)\, dp_x\, dp_y$$

where $\rho_S$ is the space density. The phase space density is a solution of the Liouville equation with the mean electric field

$$\boxed{\frac{\partial\rho}{\partial s} + [\rho, H] = 0} \qquad \int \rho\, dx\, dy\, dp_x\, dp_y = 1$$

### Uniform space distribution

The electric field is linear and the potential within the cilinder is

$$V = -\left(\omega_0^2 - \frac{\xi}{R^2}\right)\frac{x^2+y^2}{2} \qquad \boxed{\frac{\epsilon^2}{R^4} = \omega_0^2 - \frac{\xi}{R^2}}$$

**The KV distribution** Gives uniform space and momentum distributions

$$\rho = \frac{\omega}{2\pi^2\epsilon}\,\delta(H-E) \qquad\qquad E = \frac{\epsilon\omega}{2}$$

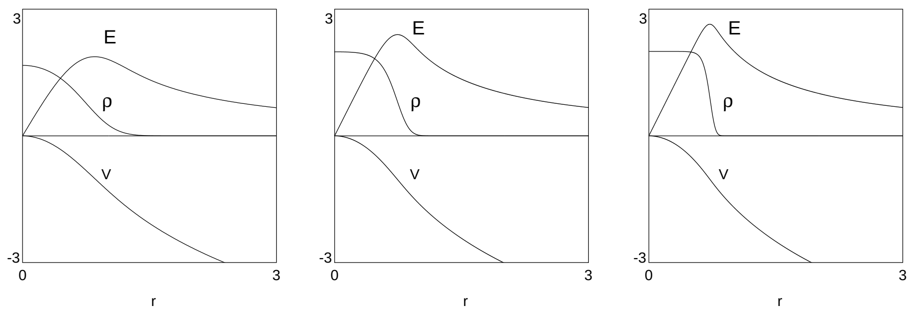
**Maxwell Boltzmann distribution**

This is is the canonical ensemble self consistent distribution

$$\rho = Z^{-1}\,\exp\Big(-\frac{p_x^2+p_y^2+\omega_0^2(x^2+y^2)+\,\xi\,V}{2\,k_B T}\Big)$$

and $V$ has to determined solving the Poisson equation with initial condition $V(0)=V'(0)=0$

$$\frac{1}{r}\frac{d}{dr}r\frac{dV}{dr} = -4\pi\rho_s(0)\,\exp\Big(-\frac{\omega_0^2 r^2+\xi V(r)}{2k_B T}\Big)$$



# 3. Simulations and collisional effects

## Integration methods

Runge Kutta of order 4 (5) with variable time step: 4 (6) evaluations.

Symplectic Runge Kutta of order 4 with constant time step: 4 evaluation.

Stoerm method with variable time step: 1 evaluation.

## Field computation

Computational complexity $C(N) \propto N^2$ reduced to

$$\boxed{C(N) \propto N \log N}$$

by multipolar expansion of far field, continued fraction reconstruction, hyerchical space splitting. Requiring $10^{-8}$ accuracy

**Low** $N$ If $N \le 10^3$ direct computations convenient: $N = 10^3$ time 0.02 s.

**High** $N$ For optimal $N > 10^3$ Linear rise : $N = 10^5$ time 2 s.

## Running times $\xi = 1$

**Low resolution:** $N = 100$ steps/cell as Vlasov ( $\Delta s = 1$ cm )

Hard collisions unresolved.

$N = 10^3$ $t_{\text{CPU}} = 2$ sec/cell $t_{\text{CPU}} = 1$ hour up to relax. ( $s_{\text{rel}} = 1500$ )

$N = 10^4$ $t_{\text{CPU}} = 20$ sec/cell $t_{\text{CPU}} = 4$ days up to relax. ( $s_{\text{rel}} = 15000$ )

$N = 10^5$ $t_{\text{CPU}} = 200$ sec/cell $t_{\text{CPU}} = 1$ year up to relax. ( $s_{\text{rel}} = 150000$ )

**High resolution:** $N = 10000$ steps/cell ( $\Delta s = 0.1$ mm )

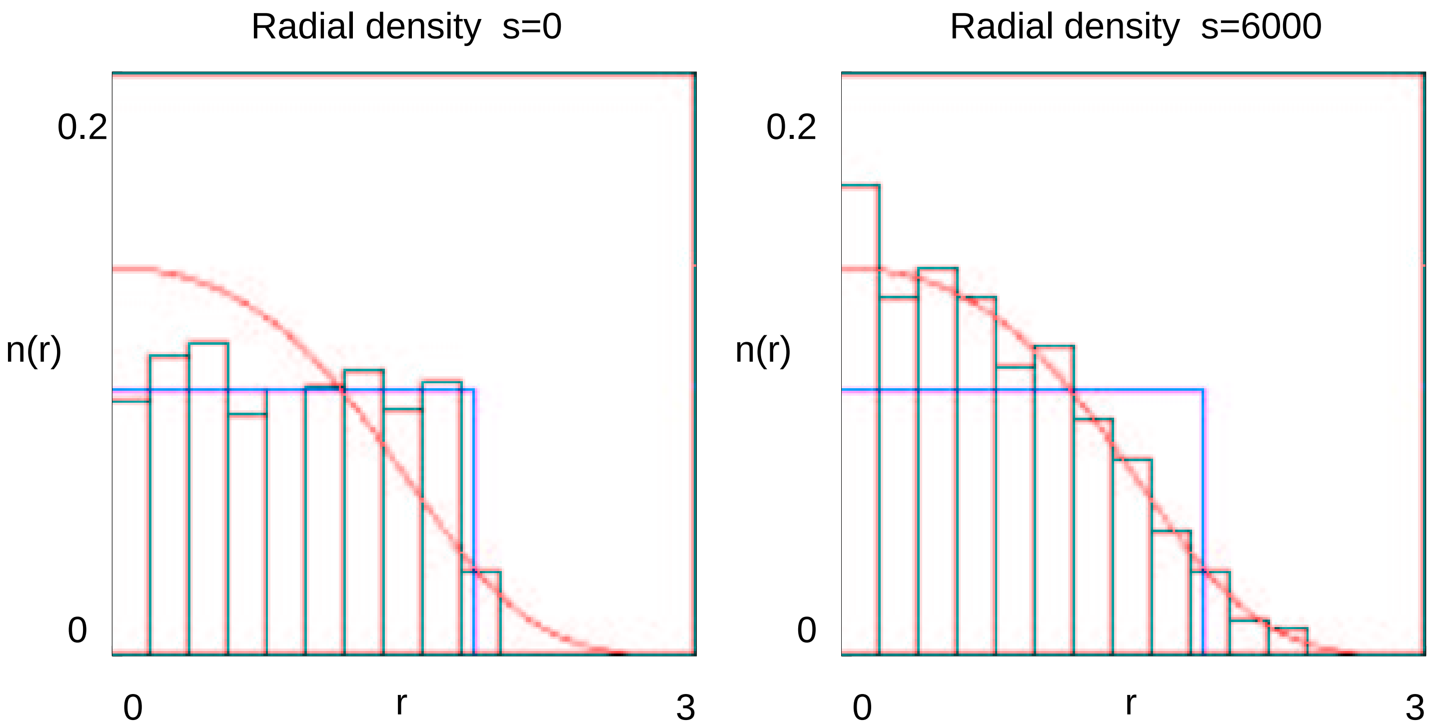
Hard collisions resolved. $t_{\text{CPU}}$ increased by 100

## Initial distribution

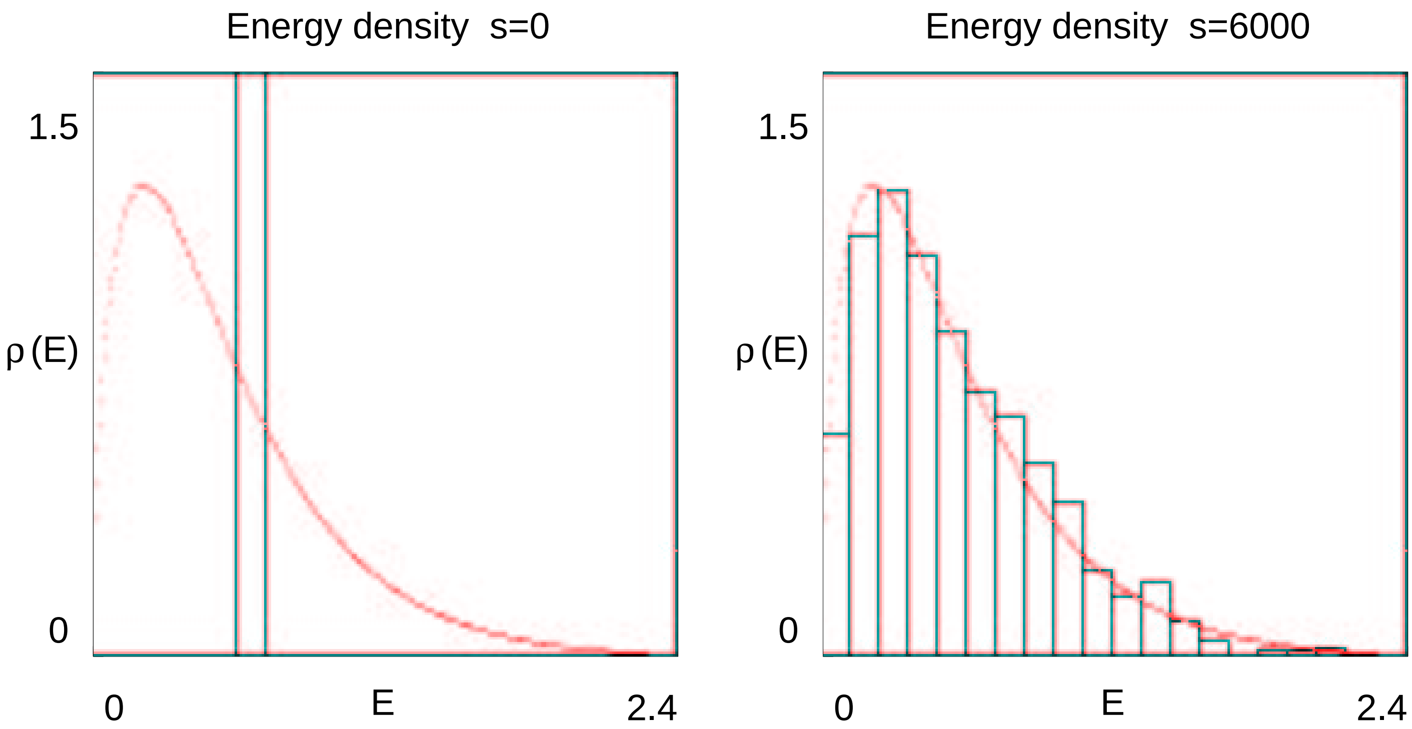
Was chosen as a matched or sightly mismatched KV

## Check-points

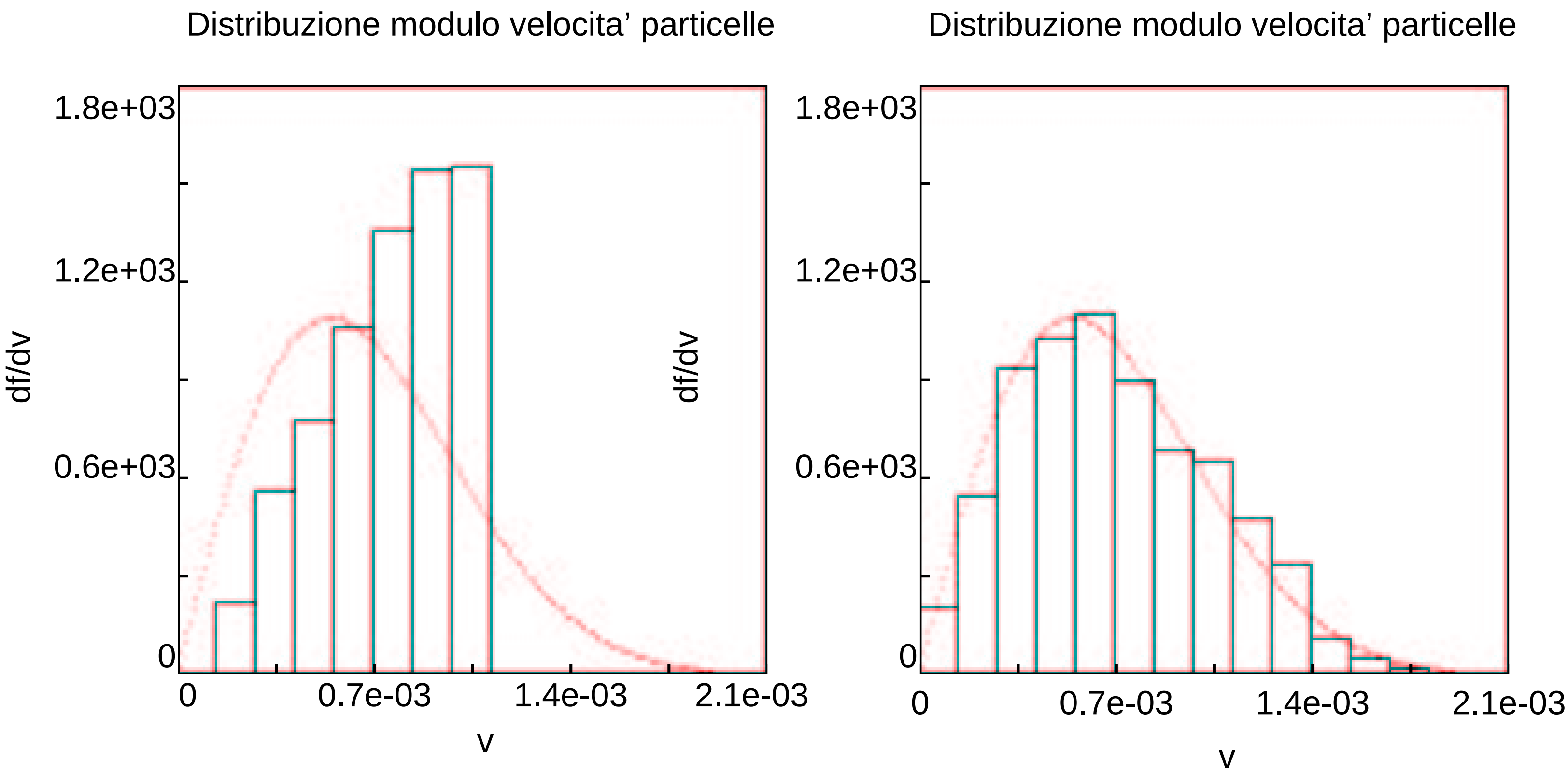
Conservation of total energy and angular momentum. Relaxation to a self consistent Maxwell-Boltzmann distribution. Thermalization of horizontal and vertical temperatures. Landau's kinetic theory comparison.



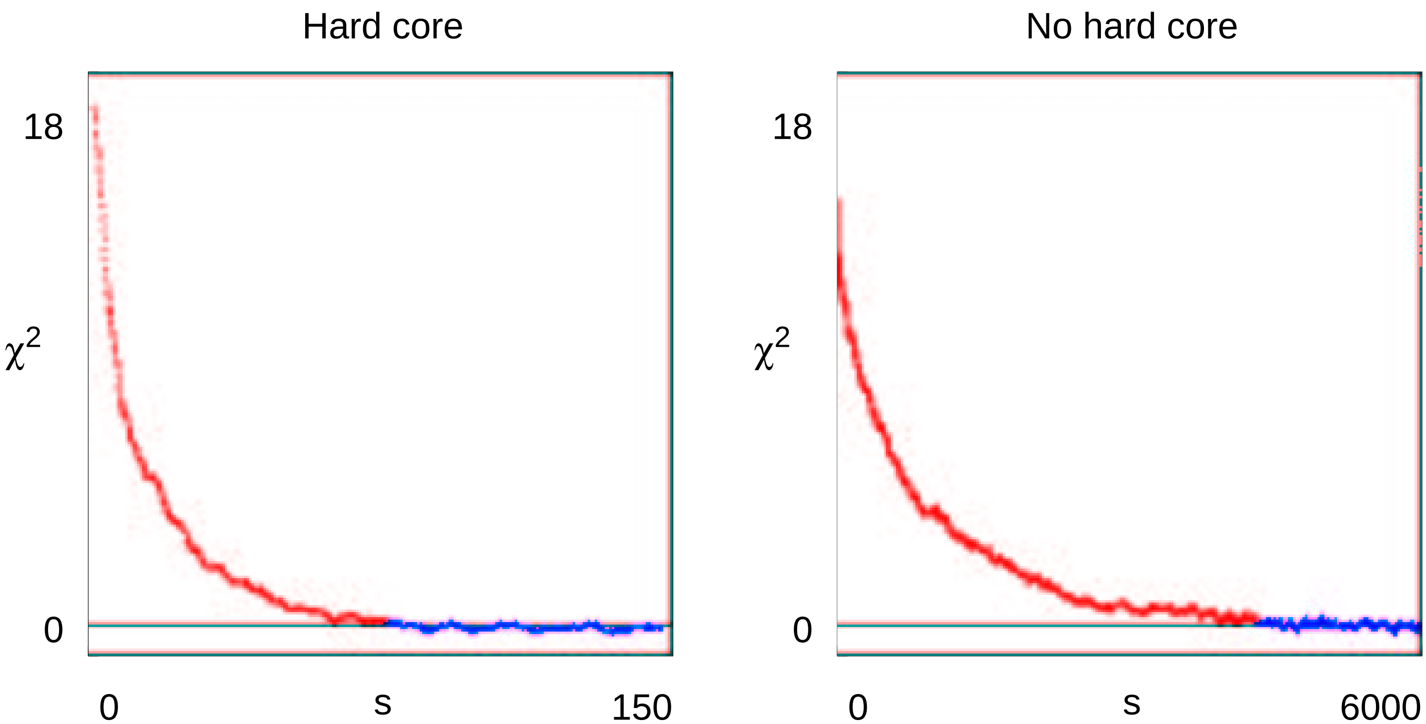

**Space density for a symmetric focusing** for particles without hard core at $s=0$ (left) and $s=6000$ (right). The parameters are $\omega_0=1$ (rad/m), $\xi=2$ and $\epsilon=1.08$ (mm mrad). The core radius is $R=1.84$ mm. so that $\lambda_D=R/5=0.382$ mm. The red curve is the self consistent Maxwell Boltzmann distribution



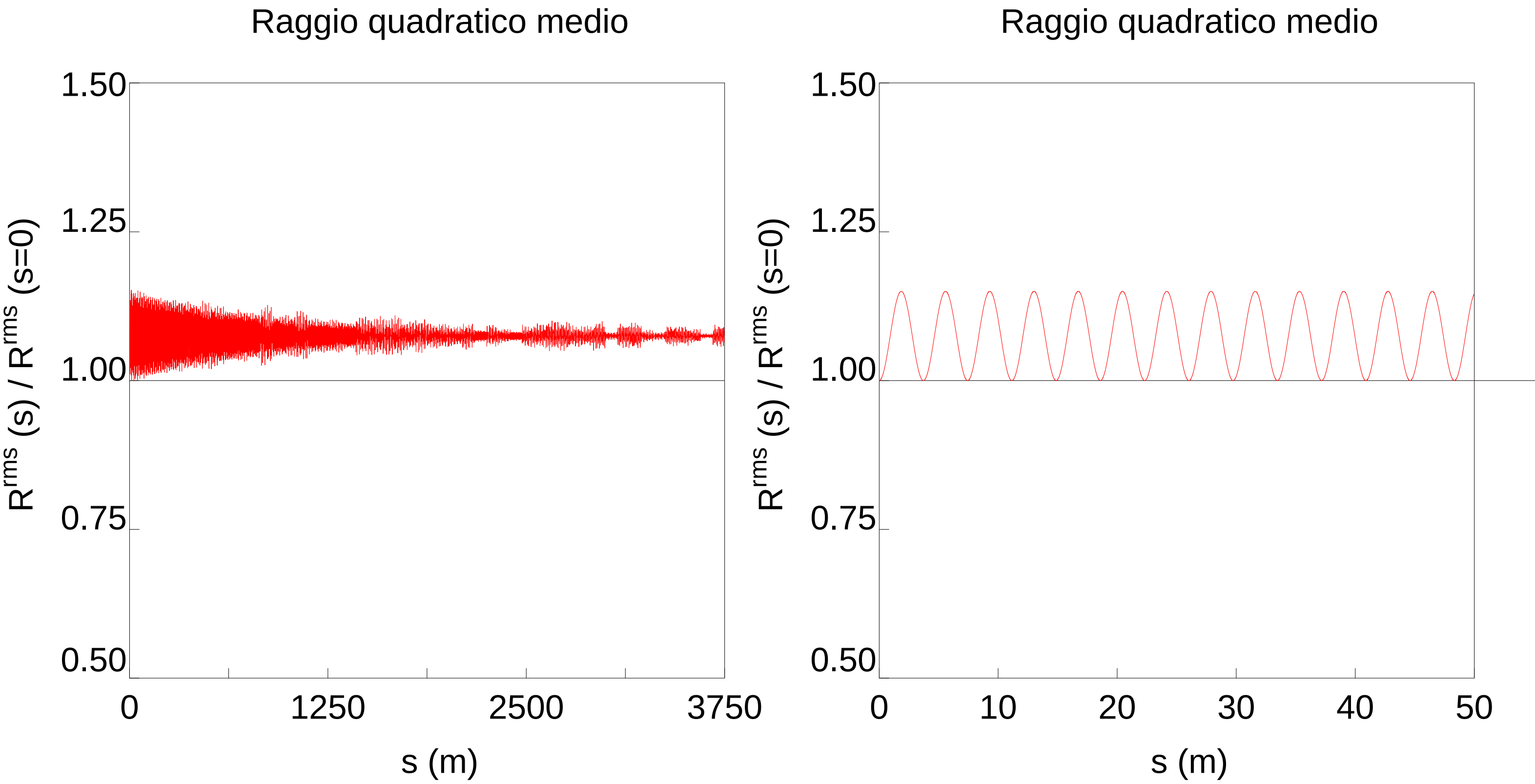

**Energy density** for particles without hard core at $s=0$ (left) and $s=6000$ (right). The red curve is the self consistent Maxwell Boltzmann distribution



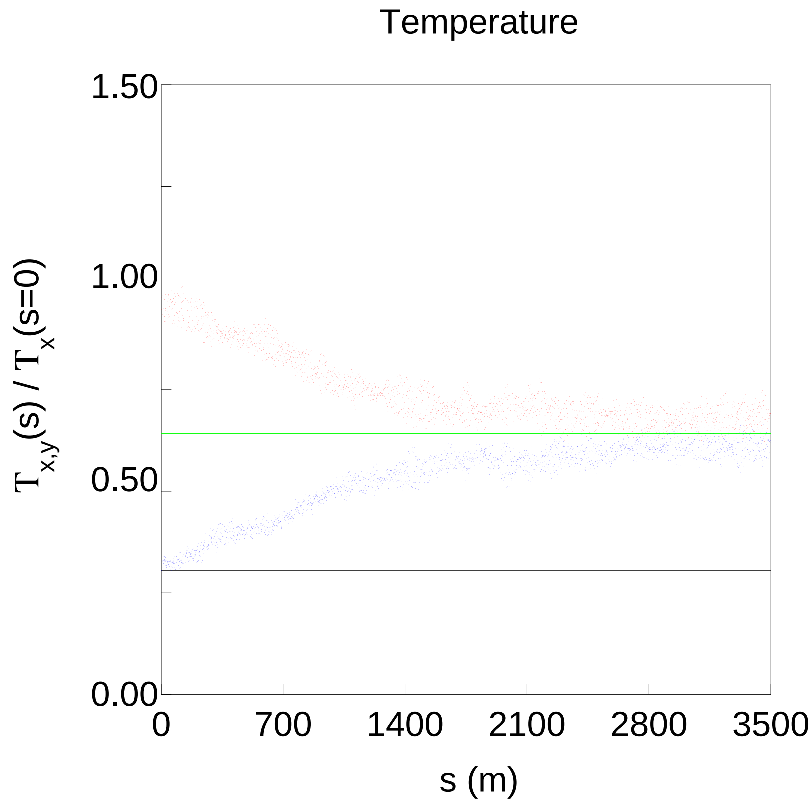

**Momentum density** for particles without hard core at $s=0$ (left) and $s=6000$ (right). The red curve is the self consistent Maxwell Boltzmann distribution




**Relaxation to equilibrium** The $\chi^2$ for the discrepancy of the phase space distribution $\rho$ with respect to the equilibrium Maxwell Boltzmann distribution is shown for the case with a hard core (left frame) and without hard core (right frame)




**Mismatch** The $\chi^2$ for the discrepancy of the phase space distribution $\rho$ with respect to the equilibrium Maxwell Boltzmann distribution is shown for the case with a hard core (left frame) and without hard core (right frame)



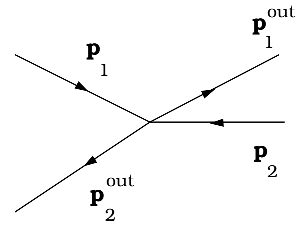

**Thermalization for asymmetric focusing:** Bare phase advances $\omega_{0\,x}=0.9$, $\omega_{0\,y}=1.1$ (rad/m). Perveance $\xi=2$ , tune depressions $\sigma_x=0.7$, $\sigma_y=0.6$ . Particles used $N=500$ , rms emittances $\epsilon_x=1$, $\epsilon_y=0.3$ (mm mrad) . Letting $\langle p_x^2 \rangle = k_B T_x$ and $\langle p_x^2 \rangle = k_B T_y$ we plot $T_x(s)/T_x(0)$, $T_y(s)/T_x(0)$ . After the relaxation $s > s_{\,\mathrm{rel}}$ one has $T_x = T_y$ .

# 4. Comparison with kinetic theory

The collisional effects can be treated by assuming that the relevant contribution comes fom binary collisions.

**Boltzmann equation** It is obtained by truncating the BBGKY hierarchy. Exact for hard spheres in the limit $N \to \infty$, $r_H \to 0$ with $r_H^2$ fixed. The kinematics of binary collisions is

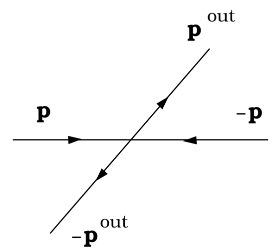
$$\mathbf{p}_1{}^{\rm out} = \mathbf{P} + \frac{\mathbf{p}^{\rm \ out}}{2} \qquad \mathbf{p}_{2\ \rm out} = \mathbf{P} - \frac{\mathbf{p}^{\rm \ out}}{2}$$



Laboratory frame

The Boltzmann equation reads

$$\boxed{\frac{\partial \rho}{\partial s} + [\rho, H] = J(\mathbf{r}, \mathbf{p}, t)}$$



Center of mass frame

where the collision integral is

$$J(\mathbf{r}, \mathbf{p}_2, t) = N \int \|\mathbf{p}_2 - \mathbf{p}_1\| \frac{d\sigma}{d\theta} \left[\rho(\mathbf{r}, \mathbf{p}_1{}^{\rm out})\, \rho(\mathbf{r}, \mathbf{p}_2{}^{\rm out}) - \rho(\mathbf{r}, \mathbf{p}_1)\, \rho(\mathbf{r}, \mathbf{p}_2)\right] d\theta\, d\mathbf{p}_1$$

# Landau's equations

Assuming the binary collisions are small angle frequent and istantaneous thay can be treated as a random process.

Let the changes of position and momentum in the interval $\Delta s$ be given by

$$\Delta \mathbf{x} = \frac{\partial H}{\partial \mathbf{p}} \Delta s \qquad \Delta \mathbf{p} = -\frac{\partial H}{\partial \mathbf{r}} \Delta s + (\Delta \mathbf{p})_c$$

collisions

The master equation reads

$$\boxed{\frac{\partial \rho}{\partial s} + [\rho,\, H] = -\sum_i \frac{\partial}{\partial p_i}(K_i\,\rho) + \frac{1}{2}\sum_{i,j=1}^{2} \frac{\partial^2}{\partial p_i \partial p_j}(D_{ij}\,\rho)}$$

where $H$ is the mean field Hamiltonian

$$H = \frac{p_x^2 + p_y^2}{2} + \omega_0^2 \frac{x^2 + y^2}{2} + \frac{\xi}{2} V(x,y) \qquad \Delta V = -4\pi \int \rho \, d\mathbf{p}$$

and the r.h.s. is the contribution of collisions

$$\mathbf{K} = \left\langle \frac{\Delta \mathbf{p}}{\Delta s} \right\rangle \qquad D_{ik} = \left\langle \frac{\Delta p_i \Delta p_k}{\Delta s} \right\rangle$$

## Final result

After integrating over the center of mass scattering angle $\Theta$ we obtain

$$\boxed{\mathbf{K}(\mathbf{p}_2) = -\beta\,\mathbf{p}_2}$$

where $\beta$ is a positive coefficient

$$\beta = \frac{N}{2}\int_0^\infty dp_1^2\,dr^2\,\Big(\pi^2\,\rho(p_1^2, r^2, s)\Big)\,\frac{1}{2\pi}\int_0^{2\pi} d\phi\,p\,\Big(\sigma_0(p) - \sigma_1(p)\Big)\,\Big(1 - \frac{p_1}{p_2}\cos\phi\Big)$$

The momentum is $p = \left(p_1^2 + p_2^2 - 2p_1p_2\,\cos\phi\right)^{1/2}$ and the distributions are

$$\pi^2\,\rho(p^2, r^2) = \begin{cases} \frac{\omega}{\epsilon}\,\delta(p^2 - \omega^2 r^2 - \omega\epsilon) & \text{KV} \\ \dfrac{\exp\Big(-\dfrac{p^2}{2k_BT}\Big)}{2k_BT}\;\dfrac{\exp\Big(-\dfrac{\omega_0^2\,r^2 + \xi\,V(r)}{2k_BT}\Big)}{Z_r} & \text{MB} \end{cases}$$

where we have $k_BT = \omega\epsilon/4$ .

**Cross section** It is computed for the cutoff potential

$$V(r) = -\frac{\xi}{N} \log \frac{r}{\Lambda}\, \vartheta(\Lambda - r)$$

The asymptotic expression of the cross section valid when $\xi/NE \ll 1$ is

$$\boxed{\frac{d\sigma}{d\Theta} = 2\Lambda \frac{\Theta}{\Theta^2_{\max}} \vartheta(\Theta - \Theta_{\max})} \qquad \boxed{\Theta_{\max} = \frac{\xi}{NE} \frac{1}{\sqrt{c}}}$$

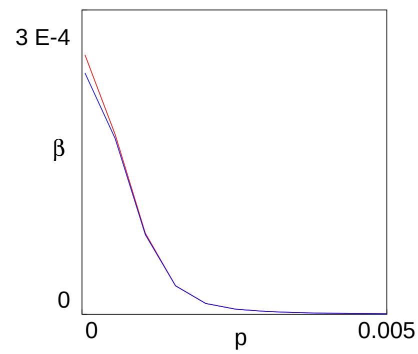
where $E = p^2$ and and $c^{-1/2} = 1.5$ , see quadratic approximation to $b(\Theta)$ . From this approximation it follows that

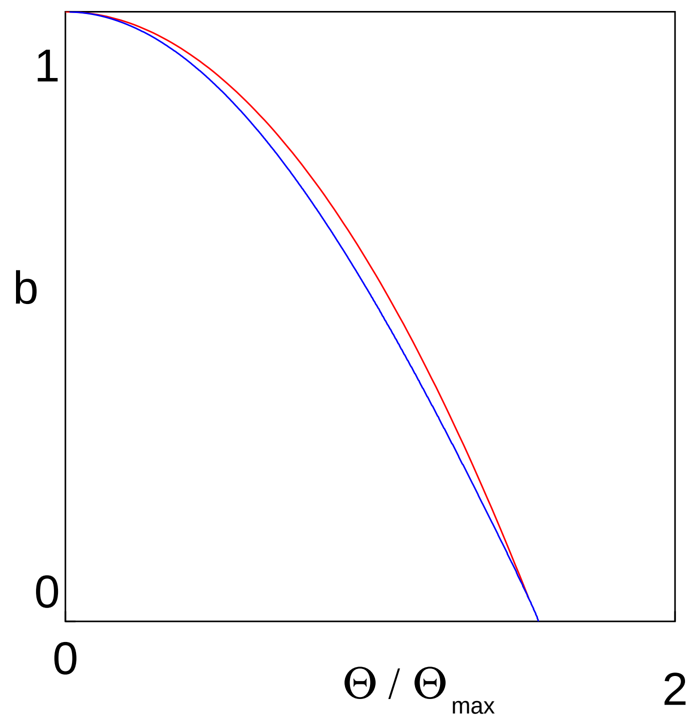
$$A(p) = \sigma_0(p) - \sigma_1(p) = \int_0^{2\pi} (1 - \cos\Theta)\sigma(\Theta, p)\, d\Theta \simeq \frac{2\Lambda}{\Theta^2_{\max}} \int_0^{2\pi} \frac{1}{2}\Theta^2\, d\Theta = \frac{\Lambda}{4}\, \Theta^2_{\max}$$

**Asymptotic drift**

The asymptotic drift is an average on $N\sigma$ the cross section and for $\xi/N \to 0$

$$\boxed{\beta = \beta_* \frac{\xi^2}{N}} \qquad \beta_{\text{phys}} = \beta \frac{N}{N_{\text{phys}}}$$

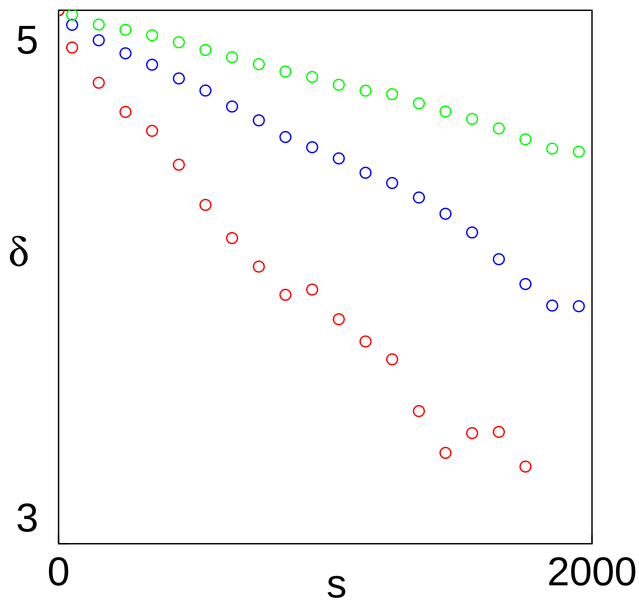





Blue curve: impact parameter $b$ versus $\Theta/\Theta_{\rm max}$ . Red curve: parabolic approximation $b=1-c(\Theta/\Theta_{\rm max})^2$ where $1/\sqrt{c}=1.52$ . The parameters are $E=0.1,\ \xi=1,\ N=10^3$ so that $\xi/(NE)=10^{-4}$ .

**Relaxation** Assuming that $D(s) = e^{A-\lambda s}$ the linear fit to $\delta(s) = \log D(s)$

$\lambda = 2.5 \times 10^{-4} \quad N = 4000 \qquad \lambda = 5 \times 10^{-4} \quad N = 2000 \qquad \lambda = 10^{-3} \quad N = 1000$




Decay of $\delta(s) = \log D(s)$ where $D$ is the distance in $L^2$ norm of the distribution $\rho(p,s)$ from the asymptotic Maxwell-Boltzmann distribution for the same perveance $\xi = 1$ bare tune $\omega_0 = 1$ and particles number $N = 1000$ red, $N = 2000$ blue, $N = 4000$ green. Fast integration.

# Comparison with simulations

The relaxation time is estimated from the Langevin equation with averaged drift and diffusion coefficients.

$$\dot{x}_i = p_i \qquad \dot{p}_i = -\omega^2 x_i - \langle\beta\rangle p_i + \langle D\rangle^{1/2}\frac{dw}{ds}$$

Justified in early stage when $\rho \simeq \rho_{\,\mathrm{KV}}$

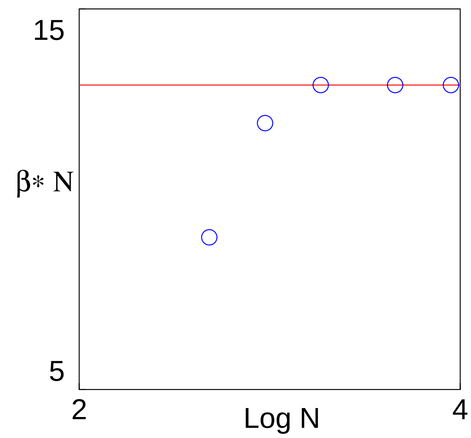
$$\left\langle (x - \langle x\rangle)^2 \right\rangle = \frac{\langle D\rangle}{4\omega^2\beta}\left[1 - e^{-2\beta\, t}\left(1 + \frac{\beta}{\omega}\sin(\omega t)\right)\right]$$

Plot $\lambda_*$ : cutoff $\Lambda = R_{\,\mathrm{KV}}$
average on $r$ up to $R_{\,\mathrm{rms}} = R_{KV}/\sqrt{2}$ .

Choose $\Lambda = R_{\,\mathrm{Debye}} = 0.29\, R_{\,\mathrm{KV}}$ ,
average up $3\, R_{\,\mathrm{rms}}$

$s^{\,\mathrm{asymp}} = \frac{2}{2\beta} = \frac{5.3}{\beta_* \times 0.29} = 1500$ vs fit $1800$

initial decay $2\beta = 1.3 \times 10^{-3}$ vs fit $8 \times 10^{-4}$



# Conclusions

**The 2D model** is proposed to investigate the spilling and rise of Maxwellian tails

**Optimal algorithms** of $N \log N$ computational complexity have developed and will be parallelized.

**Relaxation** to Maxwell-Boltzmann follows an expontial law $e^{-\beta s}$ , mismatch oscillations damp and thermalization (in asymmetric case) occurs.

**Kinetic theory** developed for 2D shows that $\beta \propto \Lambda\, \xi^2 / N$ in agreement with simulations.

# Perspectives

**Collisions 2D** Detect the weight of hard and multiple collisions on test particles comparing Langevin equations and direct simulations $N = 10^4 \sim 10^5$ .

**Mean field 2D** Criterion to choose the PIC parameters and control the numerical noise.

**Ackowledgements** To Ingo Hofmann for drawing my attention to the physics of halo, to Andrea Franchi and Giuliano Franchetti for joint work and discussions.