Equipartition, emittance and halo exchanges

Jean-Michel Lagniel, Nicolas Pichoff   CEA-DAM Bruyères-le-Châtel

Subrata Nath   LANL

Didier Uriot   CEA-DSM Saclay

1- Equipartition, a fundamental concept for thermodynamical systems

2- Coupling resonances for beam with space charge

3- Reduced parameters, tune diagram and stability chart

4- Study of the coupling resonances (case of moderate tune depressions)

1- Equipartition, a fundamental concept for thermodynamical systems

Basic courses of "thermodynamics"

"Each degree of freedom in a gas has the same mean energy kT/2"

Theorem quite intuitive :

Each degree of freedom having excess energy share it with others through multiple collisions

Do not apply to degrees of freedom unable to share energy through collisions

A "degree of freedom" is not a mechanical system and as such cannot have a mean energy !

"If a system described by classical statistical mechanics
is in equilibrium at the absolute temperature T,
every independent quadratic term in its energy has a mean value equal to kT/2"

Clear limits for the validity of the equipartition theorem :

-1- The system must be relevant of the classical statistical mechanics

-2- The system must be in equilibrium at the absolute temperature T
(Maxwell-Boltzman velocity distribution)

-3- The theorem apply to "independent quadratic terms of energy",
i.e. to linear oscillations and not to nonlinear oscillations in the general case

The concept of equipartition is unfortunately often used out of its scientific base

Beam dynamics
not
Beam thermodynamics

BEAM DYNAMICS :
EMITTANCE EXCHANGES DUE TO COUPLING RESONANCES

- space-charge forces induce coupling resonances

- these space-charge induced coupling resonances are leading to emittance and halo exchanges.

(known in circular machines)

Take into account that nonlinear space-charge forces induce both tune spreads and excitation of coupling resonances

2- Coupling resonances for beam with space charge

2-1- Definition of the parameters

σ_{ox} and σ_{oy} : phase advances / unit length at zero current

σ_x and σ_y : phase advances / unit length with space charge

$\eta_x = \sigma_x / \sigma_{ox}$ and $\eta_y = \sigma_y / \sigma_{oy}$: tune depressions

$\alpha = \sigma_y / \sigma_x$: "resonance" ratio for resonances of the type $\alpha\, \sigma_x - \sigma_y = 0$
$\alpha_o = \sigma_{oy} / \sigma_{ox}$: zero current tune ratio

$\eta = a_o / b_o$: beam size ratio with a_o and b_o the matched beam rms radii in x and y directions respectively

$\varepsilon_x = a_o^2\, \sigma_x = \pi\, a_o\, a'_o$ and $\varepsilon_y = b_o^2\, \sigma_y = \pi\, b_o\, b'_o$: rms emittances
(a'_o and b'_o : matched beam rms velocities)

$T = a'^2_o / b'^2_o$: energy ratio ($T = \varepsilon_x\, \sigma_x / \varepsilon_y\, \sigma_y$)

2-2- Analytical study of the coupling resonances

Historically mainly done for circular machines

In the case of couplings induced by perturbing fields resulting from imperfections on the magnetic elements (skew quadrupoles ...), fields from solenoids (detectors, electron cooling ...), fringe field of the quadrupoles ...

Few on couplings induced by space charge in synchrotrons

Generic 2D equations of motion for a particle subject to such a coupling "force" are :

$$d^2x/ds^2 + \sigma_x^2\, x = F_{cx}(x,y,s)$$

$$d^2y/ds^2 + \sigma_y^2\, y = F_{cy}(x,y,s)$$

General form of the perturbation "force" $F_c(x,y,s)$ described by a series with terms :

$$\delta_{lmn}\, x^l\, y^m \cos(n\,\omega\, s)$$

The resonance conditions can be expressed by

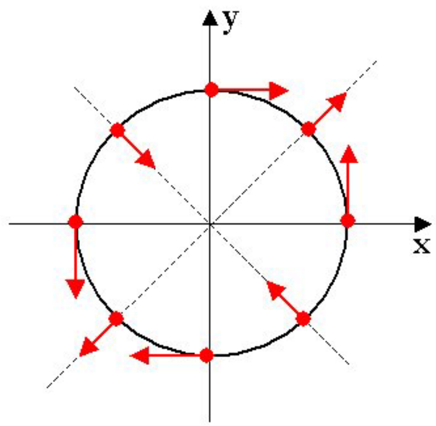
$$\mathbf{i\,\sigma_x + j\,\sigma_y = n\,\omega \quad \text{with} \quad i, j, n = 0, \pm 1, \pm 2, \pm 3, \ldots}$$

$$\mathbf{|i| + |j| \quad = \quad \text{the order of the resonance}}$$

The case of a linear coupling ($\sigma_x - \sigma_y = \omega$)

Linear perturbation force of the form :

$F_{cx}(x,y,s) = \delta\, y \cos(\omega\, s)$ $\qquad$ $F_{cy}(x,y,s) = \delta\, x \cos(\omega\, s)$



The linear coupling forces can be associated to the potential : **$V(x,y,s) = - \delta\, x\, y \cos(\omega\, s)$**

odd symmetry $\rightarrow$ coupling mode sometime called **"second order odd mode"**

The equations of motion can be integrated assuming δ small with respect to $\sigma_{x,y}{}^2$

$$d^2x/ds^2 + \sigma_x{}^2\, x = k\, y_0 \cos(\sigma_y\, s) \cos(\omega\, s)$$

$$d^2y/ds^2 + \sigma_y{}^2\, y = k\, x_0 \cos(\sigma_x\, s) \cos(\omega\, s)$$

The resonance conditions are :

$$\sigma_x - \sigma_y = \pm\, \omega \quad \textbf{and} \quad \sigma_x + \sigma_y = \omega$$

The **$\sigma_x + \sigma_y = \omega$ "sum resonance"** leads to an **unstable motion in both planes**

(unbounded particle trajectories)

With E_x and E_y the "emittances" described by the single-particle trajectories

$E_x - E_y(s) = E_x - E_y(s=0)$ remains constant

but **E_x and E_y are unbounded**

The motion is not unstable for the $\sigma_x - \sigma_y = \pm\,\omega$ "difference resonance"

beating in amplitude between the x and y motions of the particles

$E_x + E_y(s) = E_x + E_y(s=0)$ remains constant

with **exchanges between E_x and E_y** (stable but coupled motion)

NOTA : the other second order mode : $V(x,y,s) = -(\delta_x x^2 + \delta_y y^2)\cos(\omega s)$

The "second order even mode" is not a coupling mode

even if sometime considered in coupling resonance studies

It corresponds to envelope oscillations induced by a mismatch

"breathing mode" when δ_x and δ_y have the same sign

"quadrupolar mode" when δ_x and δ_y have opposite signs

Leading role of the half integer resonance

Higher-order coupling resonances

The equations of the trajectories are linear
just because the nonlinear terms have been neglected (H. Bruck)

$$V(x,y,s) = -V_0 \, \Sigma_{ij} \, (k_{ij} \, x^i \, y^j) \cos(\omega \, s)$$

Second, third and fourth order coupling resonances
associated with a given coefficient of the perturbing potential for $\omega = 0$

V coefficients	Excited coupling resonances	$\alpha = \sigma_y / \sigma_x$	
$k_{11} \, x \, y$	$\sigma_x - \sigma_y = 0$	$\alpha = 1$	2nd order odd
$k_{12} \, x \, y^2$	$\sigma_x - 2 \, \sigma_y = 0$	$\alpha = 1/2$	3rd order even
$k_{21} \, x^2 \, y$	$2 \, \sigma_x - \sigma_y = 0$	$\alpha = 2$	3rd order odd
$k_{13} \, x \, y^3$	$\sigma_x - 3 \, \sigma_y = 0$	$\alpha = 1/3$	4th order
$k_{31} \, x^3 \, y$	$3 \, \sigma_x - \sigma_y = 0$	$\alpha = 3$	4th order
$k_{22} \, x^2 \, y^2$	$2 \, \sigma_x - 2 \, \sigma_y = 0$	$\alpha = 1$	4th order

Non-linear couplings induced by the space-charge forces

B.W. Montague, "Fourth-order coupling resonance excited by space-charge forces in a synchrotron"
CERN 68-38, October 1968 (CERN-PS)

J-L Laclare, G Leleux et A Tkatchenko, "Non linéarités de charge d'espace"
Laboratoire National Saturne, GOC-GERMA, 74.166/TP24, décembre 1974 (SATURNE II)

Even density distributions ➔ space-charge "perturbing" potential =

$$V(x,y) = V_0 \left(k_{20}\, x^2 + k_{02}\, y^2 + k_{40}\, x^4 + k_{04}\, y^4 + k_{22}\, x^2 y^2 + \ldots \right)$$

Example : the space-charge potential resulting of a Gaussian distribution :

$V_0 = -\rho_0\, a\, b / (4\,\varepsilon_0)$ $\quad k_{20} = 2 / a(a+b)$ $\quad k_{02} = 2 / b(a+b)$

$k_{40} = -(2a+b) / 3a^3(a+b)^2$ $\quad k_{04} = -(2b+a) / 3b^3(a+b)^2$ $\quad k_{22} = -2 / ab\,(a+b)^2$

a and b = semi-axes of the elliptical cross-section

The space-charge potential is limited to the fourth order
(higher order have negligible effects)

The total perturbation Hamiltonian can be spilt into 2 terms :

$$\Delta H = \Delta H_0 + \Delta H_1$$

with $\Delta H_0 = k_{20}\, x^2 + k_{02}\, y^2$ and $\Delta H_1 = k_{40}\, x^4 + k_{04}\, y^4 + k_{22}\, x^2 y^2$

- Remark 1 - $\Delta H_0 = k_{20}\, x^2 + k_{02}\, y^2$ is associated to linear part of the space-charge forces induces the incoherent tune shifts $\Delta\sigma_x$ and $\Delta\sigma_y$

$\Delta H_0 = k_{20}\, E_x / 2\,\sigma_x + k_{02}\, E_y / 2\,\sigma_y$ when the high-frequency oscillating terms are eliminated.

The phase advances with space charge are then given by :

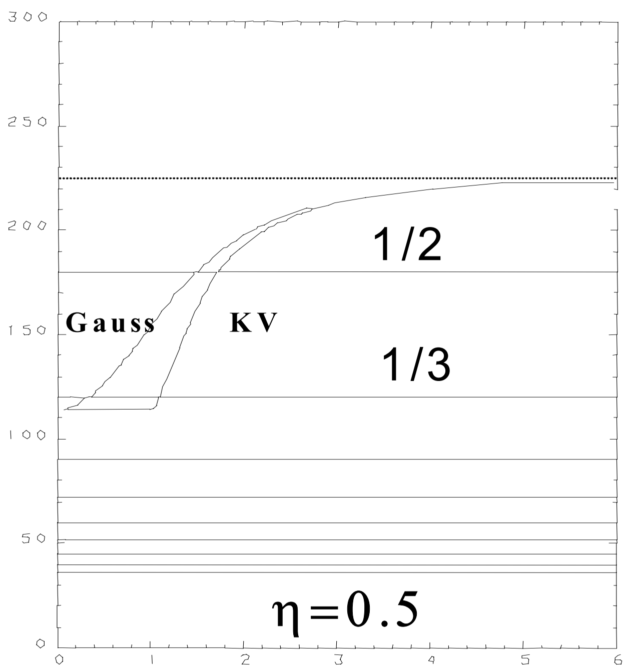
$$\sigma_x = \sigma_{0x} + \Delta\sigma_x \quad \text{with} \quad \Delta\sigma_x = \delta(\Delta H_0) / \delta E_x = k_{20} / 2\,\sigma_x \quad < 0$$

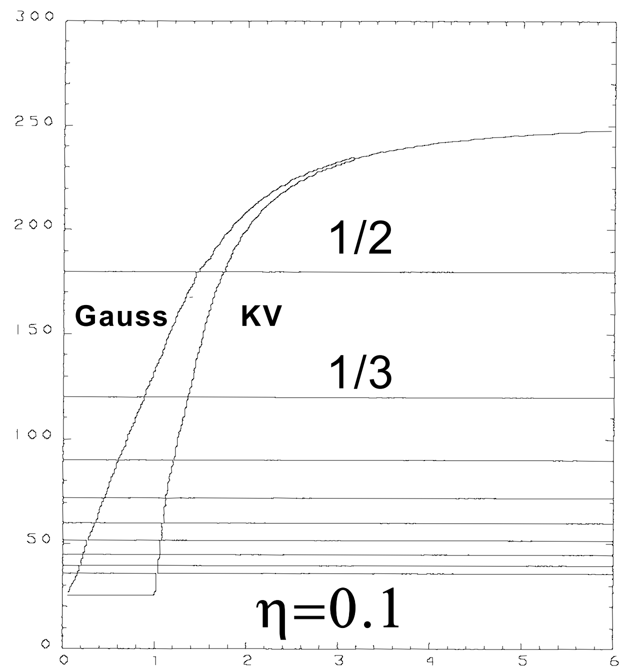
$$\sigma_y = \sigma_{0y} + \Delta\sigma_y \quad \text{with} \quad \Delta\sigma_y = \delta(\Delta H_0) / \delta E_y = k_{02} / 2\,\sigma_y \quad < 0$$

- Remark 2 - $\Delta H_1 = k_{40}\, x^4 + k_{04}\, y^4 + k_{22}\, x^2\, y^2$ gives the first nonlinear terms responsible of an amplitude dependence of the tunes :

$$\delta\sigma_x = \delta(\delta H_{cw}) / \delta E_x = A\, E_x + B\, E_y \qquad \delta\sigma_y = \delta(\delta H_{cw}) / \delta E_y = B\, E_x + C\, E_y$$

with $A = 3\, k_{40} / 4\, \sigma_x^2$ $B = k_{22} / 4\, \sigma_x\, \sigma_x$ and $C = 3\, k_{04} / 4\, \sigma_y^2$





- Remark 3 - **$k_{22}\, x^2\, y^2$ of ΔH_1 is the lowest order term of the coupling force induced by SC**

Even density distribution ➔ The higher order coupling terms have the form $k_{2n\,2m}\, x^{2n}\, y^{2m}$

They result of the non-uniform character of the charge distribution

They are always present, even for a beam of circular cross-section

V coefficients	Excited coupling resonances	$\alpha = \sigma_y / \sigma_x$	
$k_{22}\, x^2\, y^2$	**$2\sigma_x - 2\sigma_y = 0$**	**$\alpha = 1$**	**4th order**
$k_{24}\, x^2\, y^4$	$2\sigma_x - 2\sigma_y = 0$ $2\sigma_x - 4\sigma_y = 0$	$\alpha = 1/2$	6th order
$k_{42}\, x^4\, y^2$	$2\sigma_x - 2\sigma_y = 0$ $4\sigma_x - 2\sigma_y = 0$	$\alpha = 2$	6th order
$k_{26}\, x^2\, y^6$	$2\sigma_x - 2\sigma_y = 0$ $2\sigma_x - 4\sigma_y = 0$ $2\sigma_x - 6\sigma_y = 0$	$\alpha = 1/3$	8th order
$k_{62}\, x^6\, y^2$	$2\sigma_x - 2\sigma_y = 0$ $4\sigma_x - 2\sigma_y = 0$ $6\sigma_x - 2\sigma_y = 0$	$\alpha = 3$	8th order
$k_{44}\, x^4\, y^4$	$2\sigma_x - 2\sigma_y = 0$ $2\sigma_x - 4\sigma_y = 0$ $4\sigma_x - 2\sigma_y = 0$ $4\sigma_x - 4\sigma_y = 0$	$\alpha = 1$	8th order

- Remark 4 - The coupling forces are more and more determinant for the beam dynamics as the beam cross-section asymmetry increases (“flat beam”)

Example of Gaussian distribution :

Ratio of the 4th order coupling force over the linear part of the space charge force :

$$C_x = 2\, k_{22}\, y^2 / k_{20} \qquad C_y = 2\, k_{22}\, x^2 / k_{02}$$

$$\Downarrow$$

$$C_x = -b / (a + b) \qquad C_y = -a / (a+b)$$

at the position $x = a / \sqrt{2}$, $y = b / \sqrt{2}$

Independent of the beam cross-section ($S = \pi.a.b$) but strongly dependent on the asymmetry

$k = a / b$	0.1	0.2	0.5	1.0	2.0	5.0	10.
$\lvert C_x \rvert$	0.91	0.83	0.67	0.5	0.33	0.17	0.09
$\lvert C_y \rvert$	0.09	0.17	0.33	0.5	0.67	0.83	0.91

- Remark 5 – The studies are done for coupling resonances excited by the cw component of the perturbing force ($\omega = 0$)

- For a matched beam the period of envelope oscillation is the lattice the coupling resonance conditions for a particle is

$$i\,\sigma_{px} - j\,\sigma_{py} = 2\pi\, n$$

High-intensity accelerators usually designed with :

$$40° < \sigma_{0x,\,0y} < 90° \quad \text{and} \quad \eta_{x,y} > 0.5 \quad \Rightarrow \quad 20° < \sigma_{x,\,y} < 90°$$

Phase advances per focusing period of the particle in the range :

$$20° < \sigma_{px,\,py} < 90° \qquad (\,0.05 < \nu_{px,py} < 0.25\,)$$

The lowest order coupling resonance is then for $i > 5$ with $j = 2$ and $n = 1$

leading to possible excitations through resonances higher than the 7th order known to have negligible effects on the beam dynamics.

$\Downarrow$

THE FOCUSING SCHEME HAS NO EFFECT

Demonstrated by 3D PIC code simulations

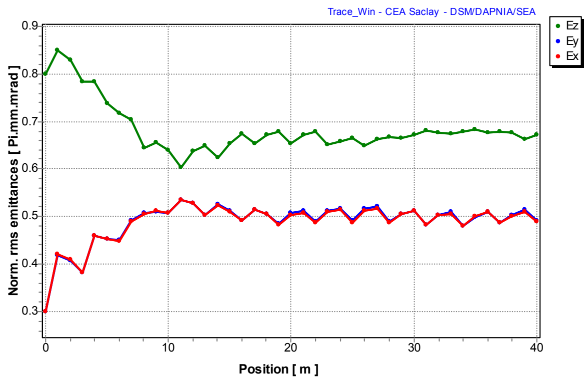
NOTA for the “sum resonances” $i\,\sigma_{px} + j\,\sigma_{py} = 2\pi\, n$ which lead to unbounded motions

the tunes are usually such that only sum resonances with orders higher than 5

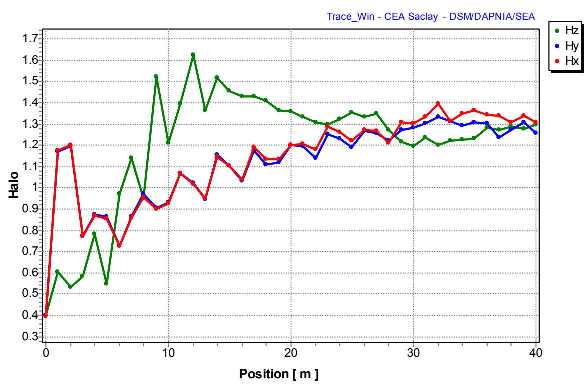
can affect the particles (long range behavior)

MATCHED BEAM, 3D beam dynamics for different focusing channels

Continuous focusing

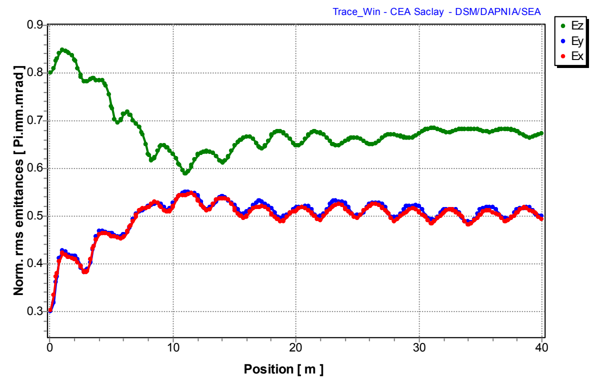


rms emittance

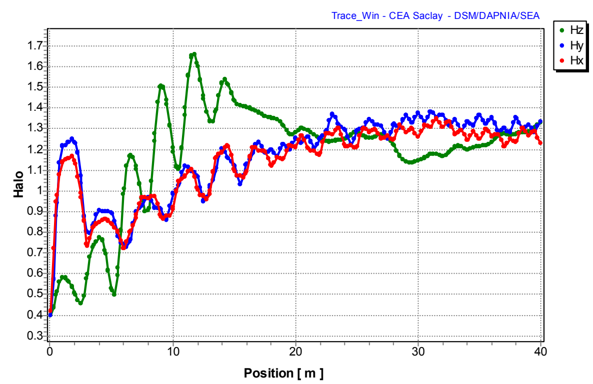


Halo

FODO

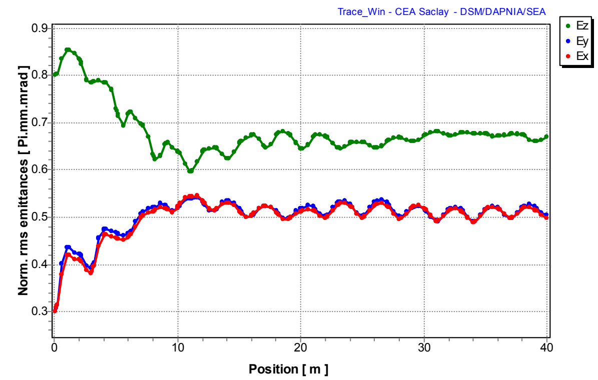


rms emittance

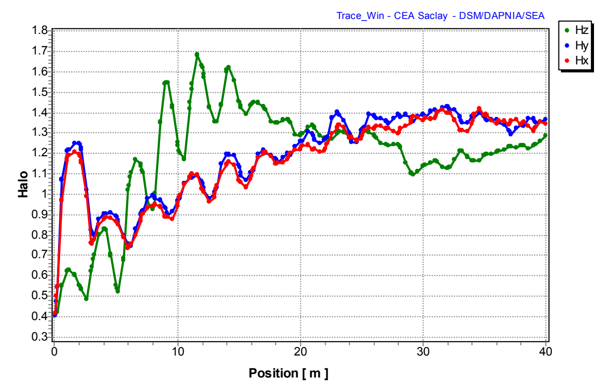


Halo

Doublet



rms emittance



Halo

- For a mismatched beam the phase advances of the envelope oscillations are given by :

$$\sigma_B^2 = 2\,(\sigma_t^2 + \sigma_{0t}^2) \quad \text{and} \quad \sigma_Q^2 = 3\,\sigma_t^2 + \sigma_{0t}^2$$

for the “breathing” and “quadrupolar” modes respectively

The resonance conditions :

$$i\,\sigma_{px} - j\,\sigma_{py} = n\,\sigma_B \quad \text{or} \quad n\,\sigma_Q$$

are fulfilled at the lowest order (j = n = 1) when :

$$i\,\sigma_{ot} - \sigma_t = [2\,(\sigma_t^2 + \sigma_{0t}^2)]^{1/2} \quad \text{or} \quad [3\,\sigma_t^2 + \sigma_{0t}^2]^{1/2}$$

then for :

$$i = \eta + [2\,(\eta^2 + 1)]^{1/2} \quad \text{or} \quad \eta + [3\,\eta^2 + 1]^{1/2}$$

then when :

i = 2.1 (Breathing) or 1.8 (Quadrupolar) for $\eta = 0.5$

MISMATCHS MUST PLAY AN IMPORTANT ROLE IN COUPLING RESONANCES WHEN $\eta < 0.6$

Look to resonances of the type $2\,\sigma_{px} - \sigma_{py} = \sigma_B$ or $= \sigma_Q$

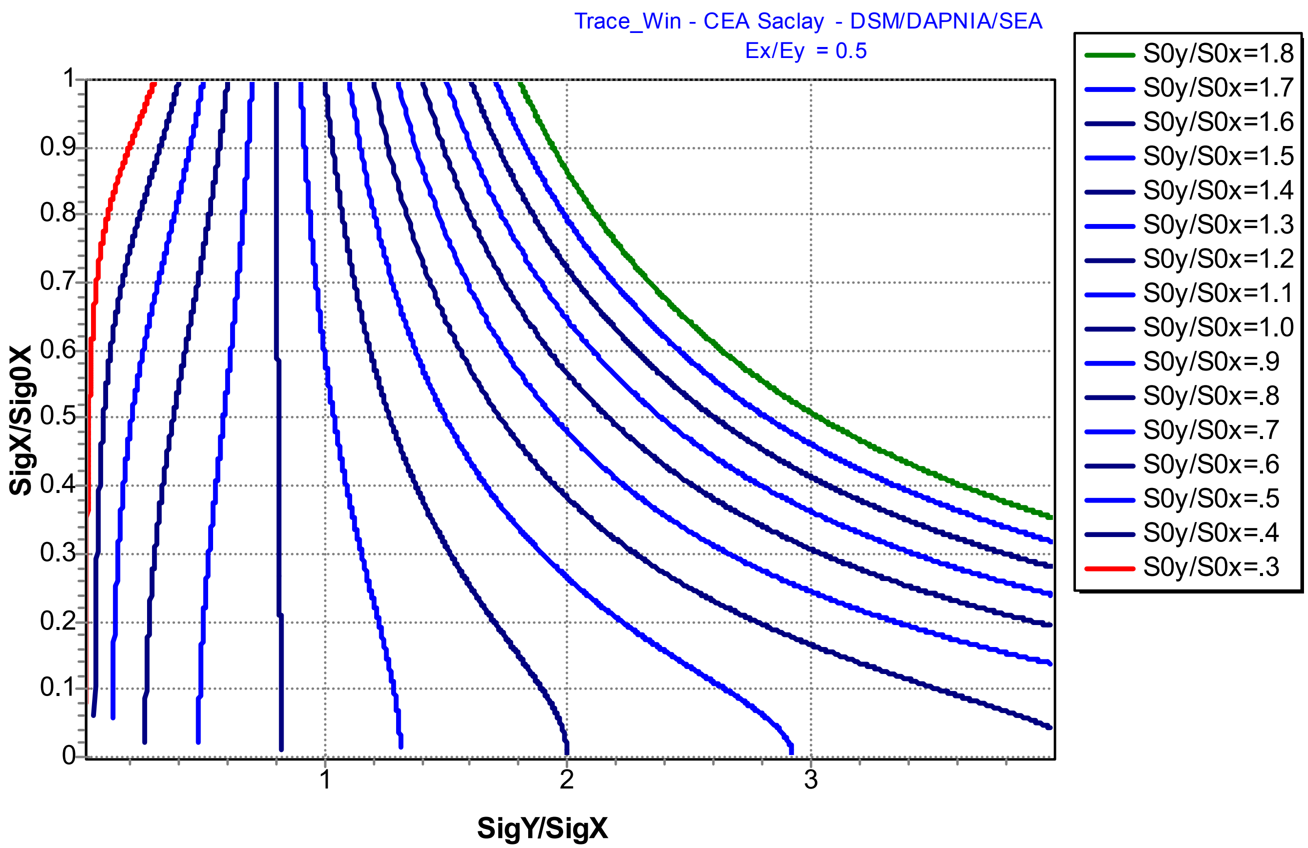
3- Reduced parameters, tune diagram and stability chart

3-1- Beam dynamics with reduced parameters

2D coupling resonances can be studied using a set of 3 reduced parameters.

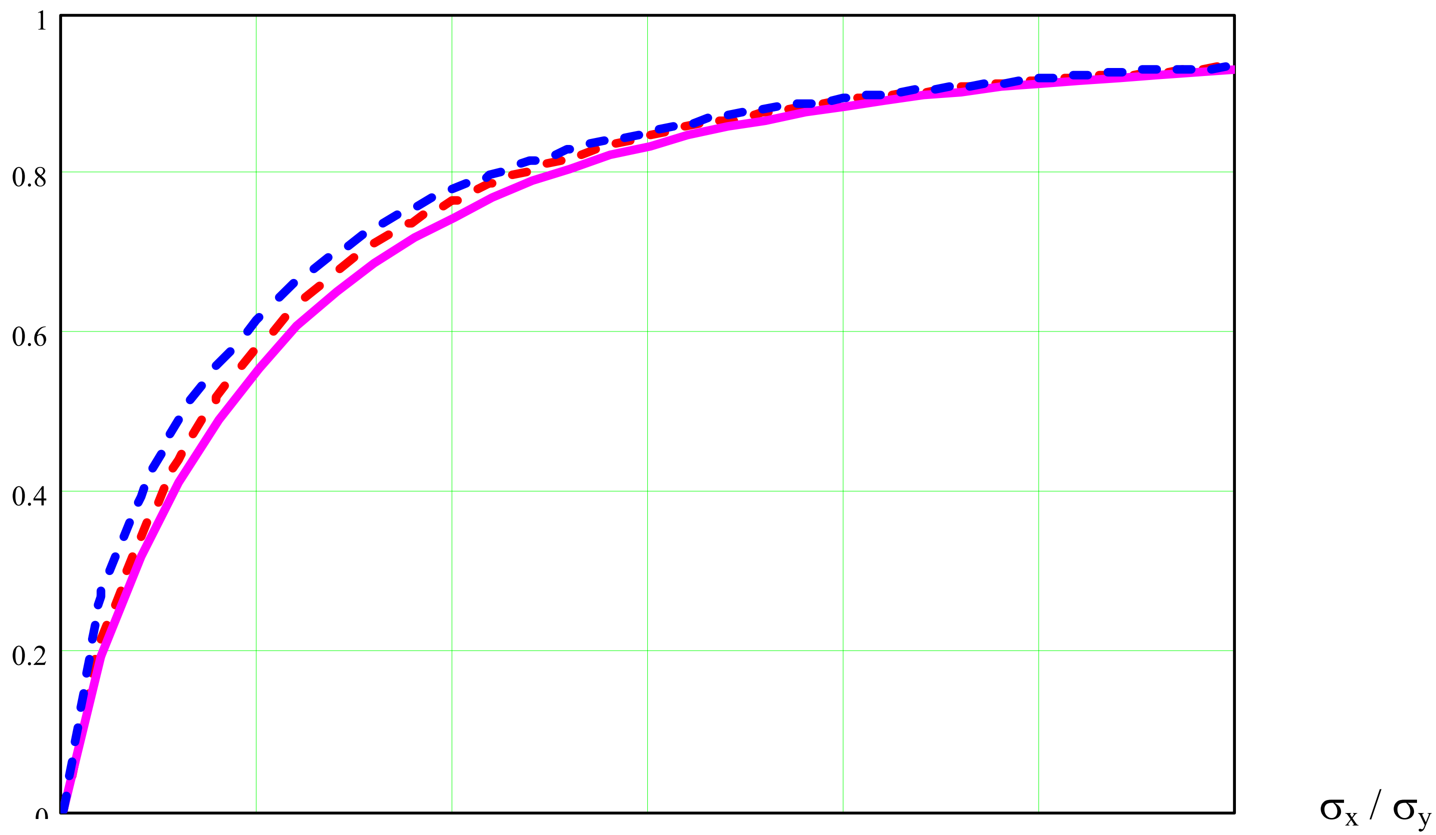
Ingo Hofmann chose : α , η_x or η_y and $\varepsilon_x / \varepsilon_y = \eta^2 / \alpha$ with $\eta^2 = \alpha\, \varepsilon_x / \varepsilon_y$

Stability charts in diagrams $(1/\alpha , \eta_y)$ or (α , η_x) for a fixed value of $\varepsilon_x / \varepsilon_y$



$\eta_x = \sigma_x / \sigma_{ox}$ as a function of $\alpha = \sigma_y / \sigma_x$ for $\sigma_{oy} / \sigma_{ox} = 0.3$ to 1.8 step 0.1 and for $\varepsilon_y / \varepsilon_x = 2$

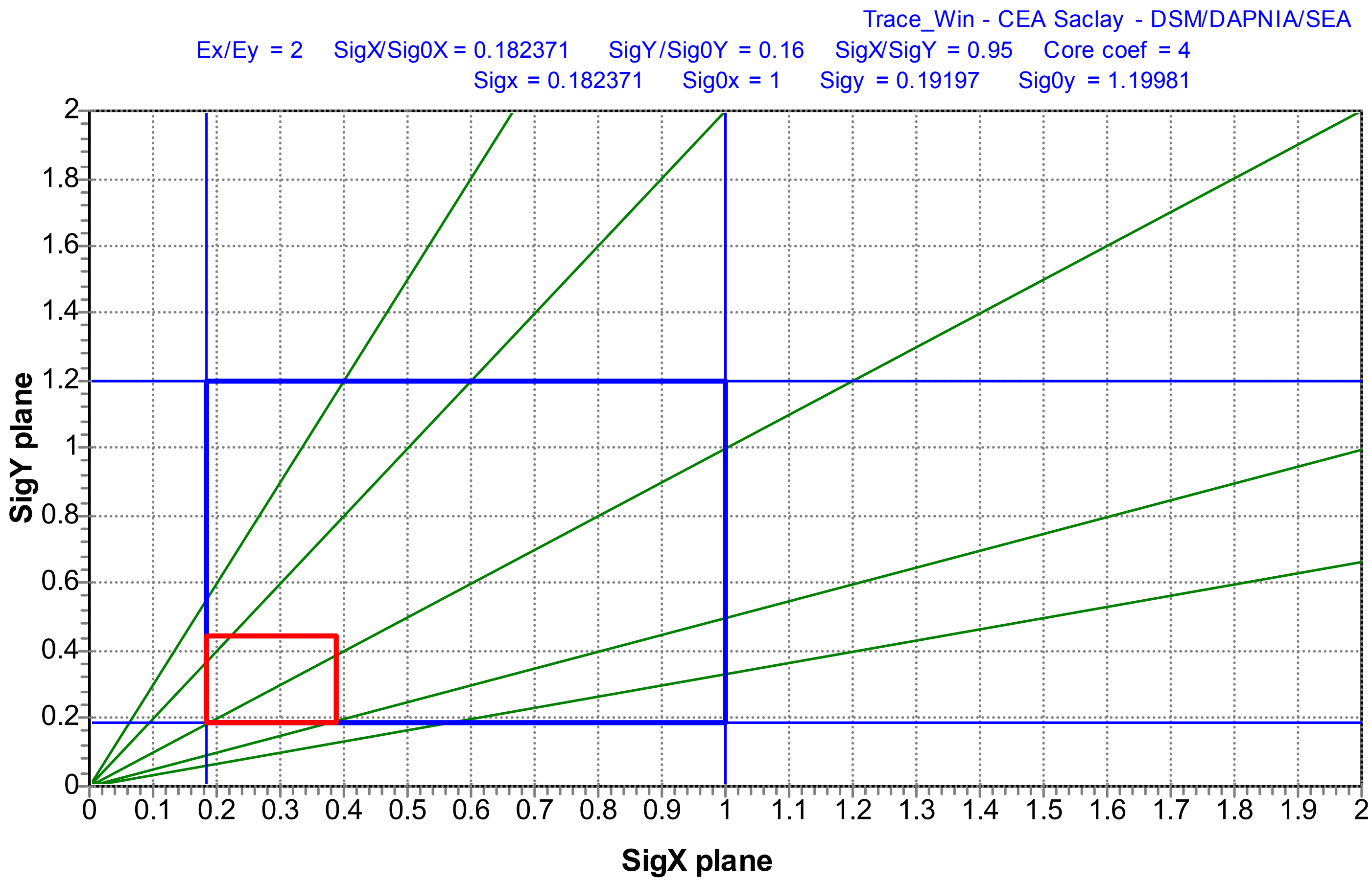
Remark : 2D studies give results close to 3D studies



η_x as a function of $1/\alpha = \sigma_x / \sigma_y$ for $\varepsilon_x / \varepsilon_y = 5$ and $\eta_y = 0.6$
2D (pink) - 3D with z = y (dashed red) and 3D with z = x (dashed blue)

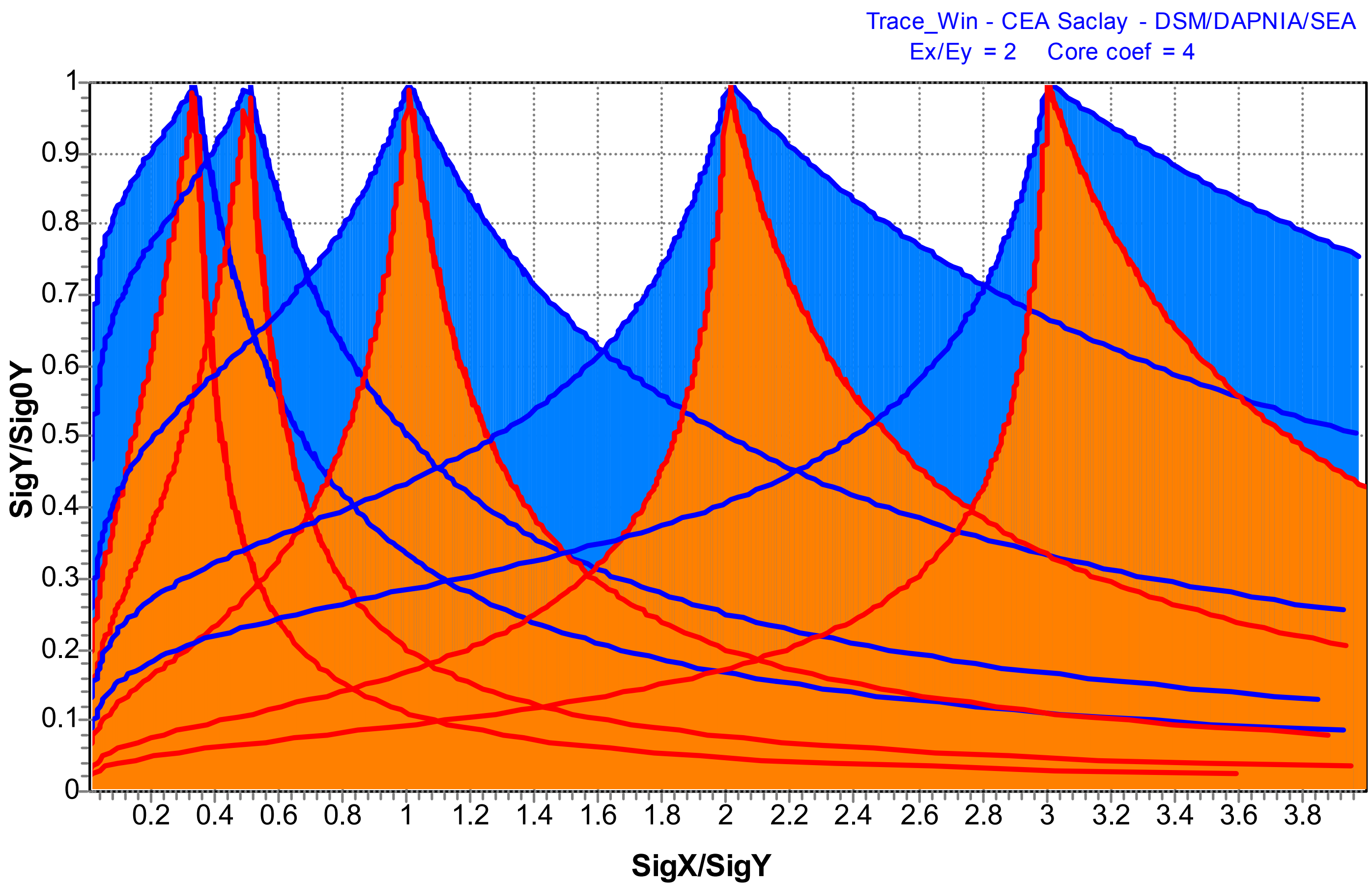
2D parameters of interest for the coupling resonance studies are weakly dependant of the 3D distribution

3-2- Tune diagrams with reduced parameters



Tune diagram for $\varepsilon_x / \varepsilon_y = 2$ $\alpha = \sigma_y / \sigma_x = 1.05$ and $\eta_y = 0.16$
leading to $\eta_x = 0.18$ $\sigma_x = 0.18$ $\sigma_{ox} = 1.00$ $\sigma_y = 0.19$ $\sigma_{oy} = 1.2$

3-3- Stability charts of the coupling resonances
(ω = 0)



Example of stability chart for the coupling resonances α = 1/3, 1/2, 1, 2 and 3

Orange = beam core affected by the resonance

Blue = halo affected by the resonance

4- Study of the coupling resonances (case of moderate tune depressions)

$\eta_x = 0.6 \quad \eta_y > 0.5$

- Values quite well representative of high-power proton linacs under study
- Sufficiently above the stochastic threshold to expect a clear identification of the coupling resonance effects and of the physics behind emittance transfers

(avoid a mixture of several different phenomena inherent to large tune depressions)

Example : Study of the $\sigma y / \sigma x = 3$ coupling resonance

Working point $\sigma_x = 0.60$, $\sigma_y = 1.764$ ➔ $\alpha = \sigma_y / \sigma_x = 2.94$ ($4^{th} - 8^{th}$ order coupling resonance)

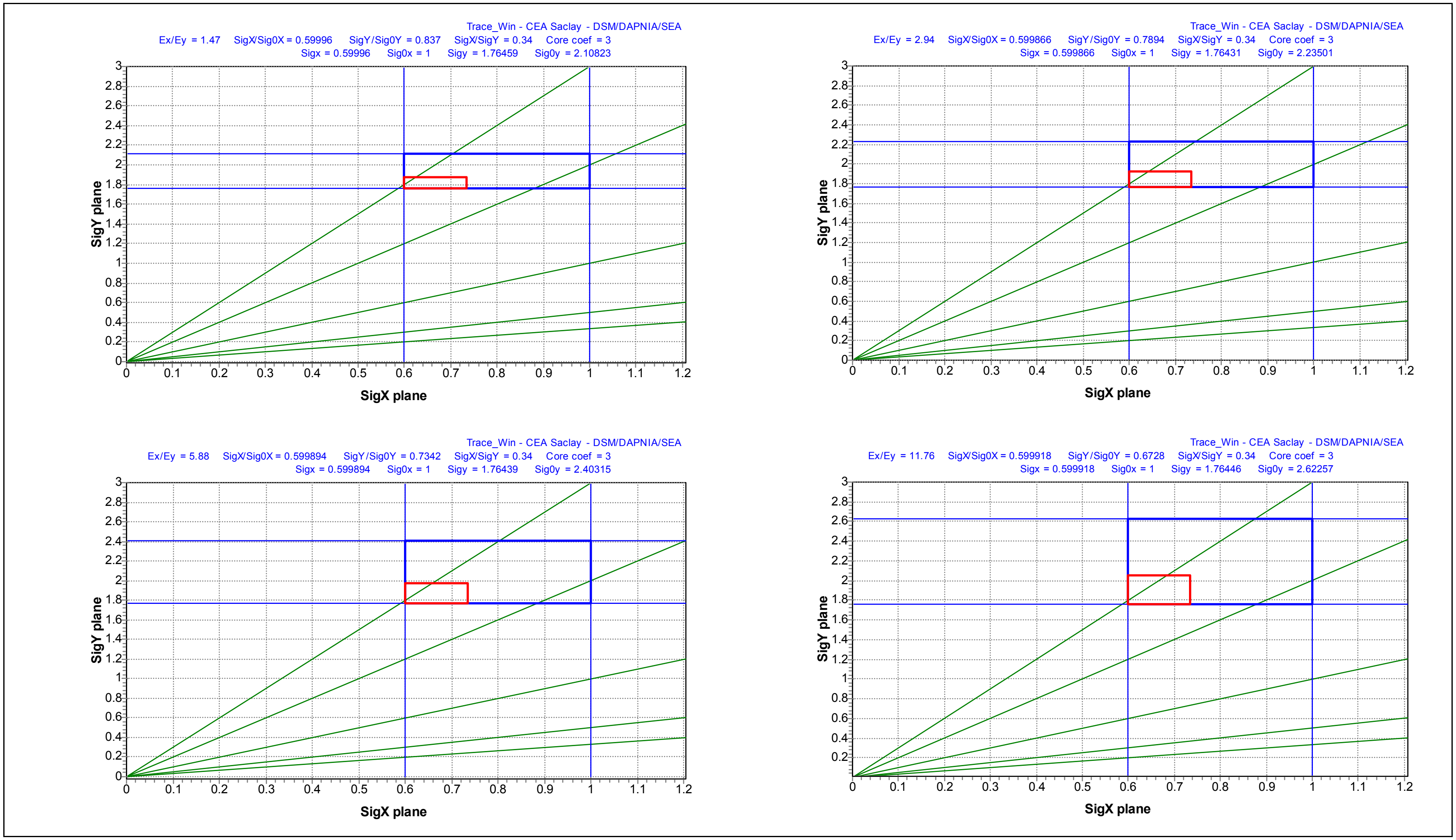
$\eta_x = 0.6$ ➔ η_y from 0.84 to 0.67

$T = a'^2_o / b'^2_o$: energy ratio ($T = \varepsilon_x \sigma_x / \varepsilon_y \sigma_y$)

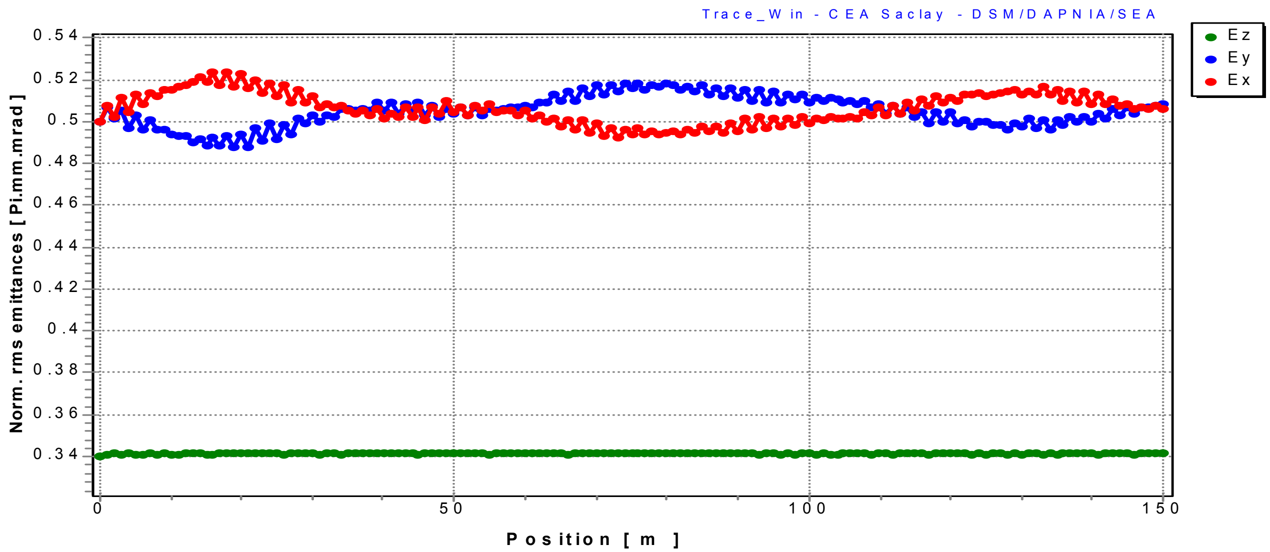
T	0.5	1.0	2.0	4.0
$\varepsilon_x / \varepsilon_y$	1.47	2.94	5.88	11.76
$\eta = a_o / b_o$	2.08	2.94	4.16	5.88
η_y	0.84	0.79	0.73	0.67
σ_{oy}	2.11	2.23	2.40	2.62

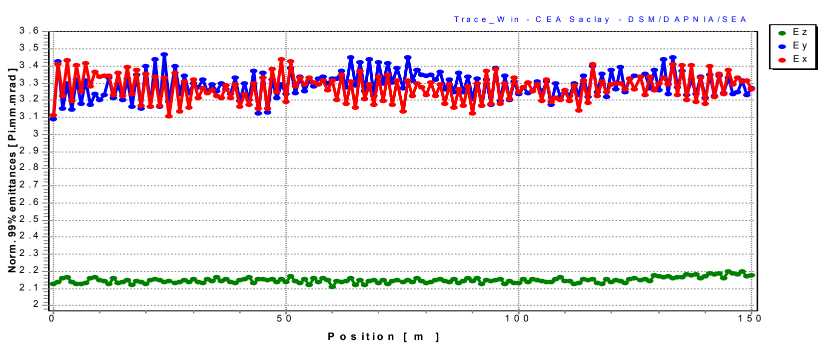
Tune diagrams

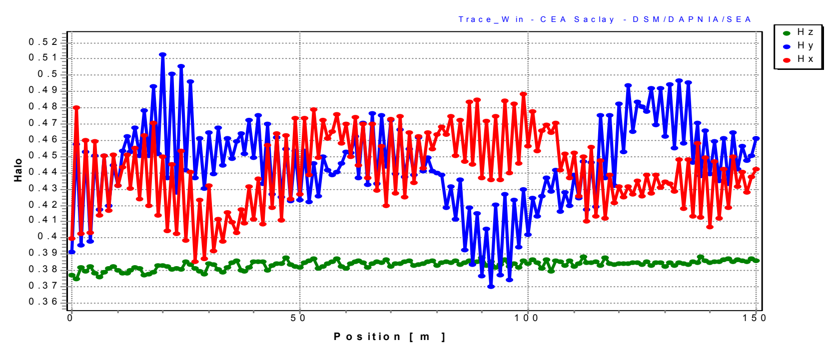
T = 0.5 T = 1.0
T = 2.0 T = 4.0



3D multi-particle simulations   Energy ratio T = 0.5



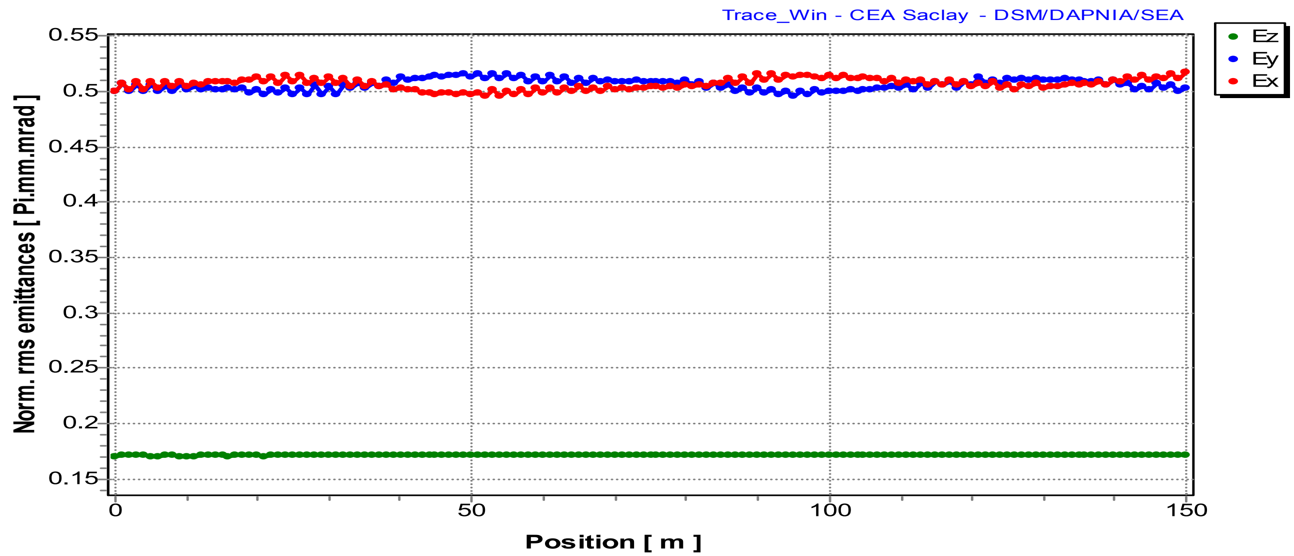


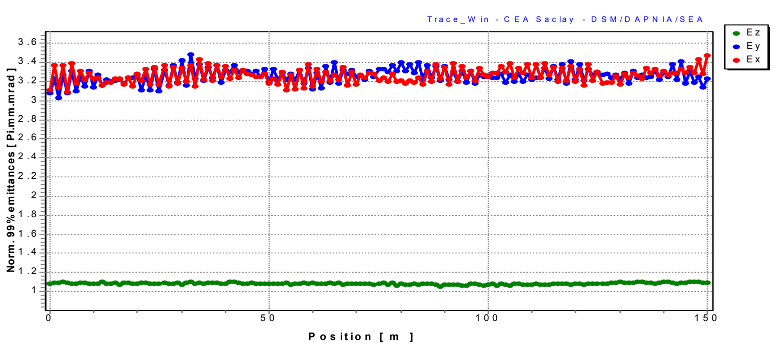


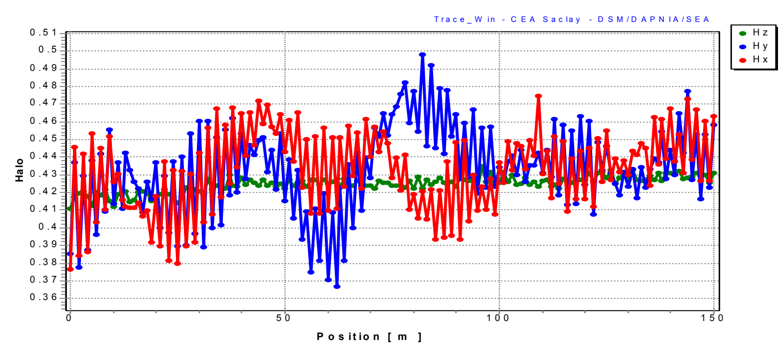
T = 0.5 $\alpha = 2.94$ $\varepsilon_x / \varepsilon_z = 1.47$ $\eta_x = 0.60$ $\sigma_{ox} = 1.0$

$\eta_z = 0.822$ $\sigma_z = 1.763$ $\sigma_{oz} = 2.145$ $\varepsilon_x = 0.5$ $\varepsilon_y = 0.34$ I = 12.7 mA

3D multi-particle simulations   Energy ratio T = 1.0



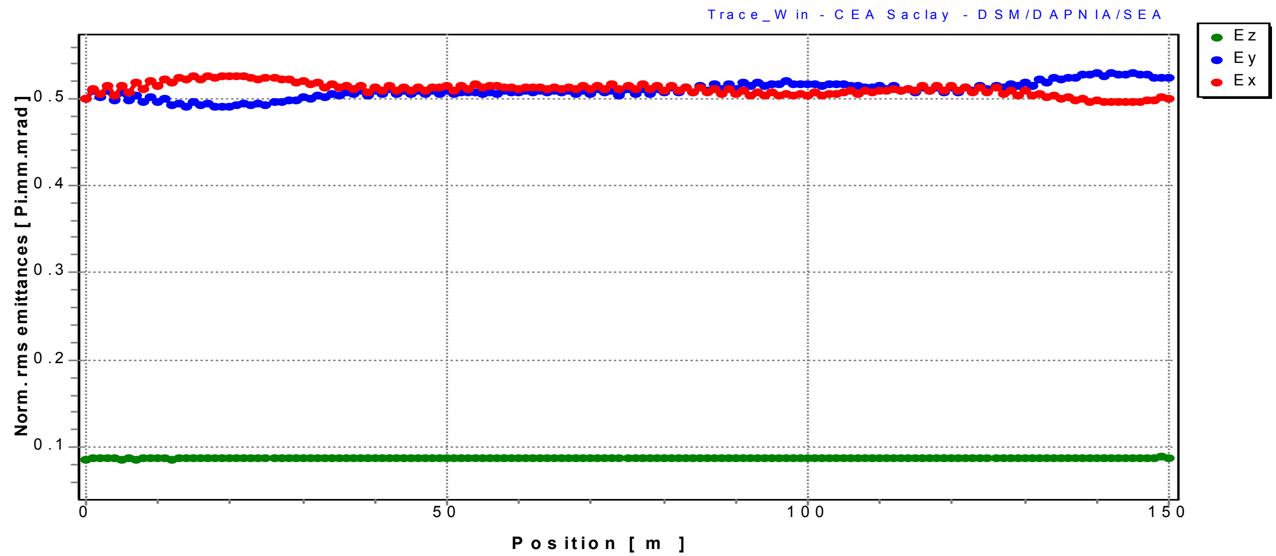


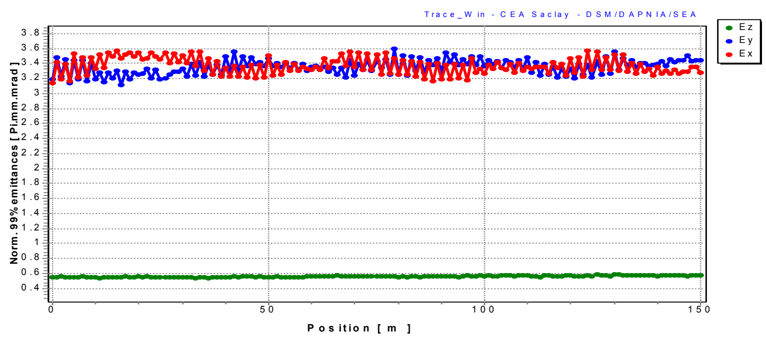


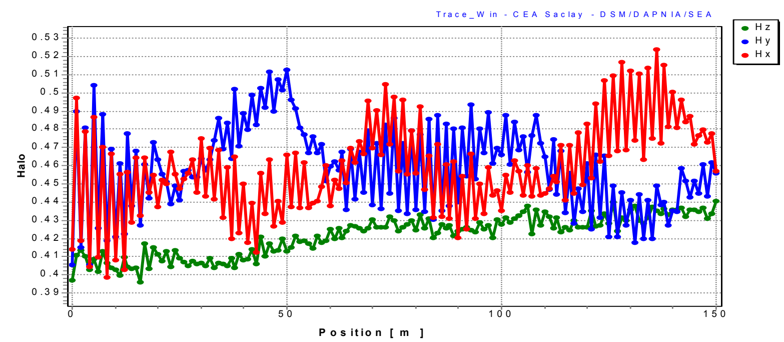
T = 1.0 $\alpha = 2.94$ $\varepsilon_x / \varepsilon_z = 2.94$ $\sigma_{oz} / \sigma_{ox} = 2.30$ $\sigma_{ox} = 1.0$ $\sigma_{oz} = 2.3$

$\eta_x = 0.60$ $\eta_z = 0.77$ $\varepsilon_x = 0.5$ $\varepsilon_y = 0.34$ I = 11.2 mA

3D multi-particle simulations   Energy ratio T = 2.0







T = 2.0 $\alpha = 2.94$ $\varepsilon_x / \varepsilon_z = 5.88$ $\sigma_{oz} / \sigma_{ox} = 2.505$ $\sigma_{ox} = 1.0$ $\sigma_{oz} = 2.505$

$\eta_x = 0.60$ $\eta_z = 0.704$ $\varepsilon_x = 0.5$ $\varepsilon_y = 0.085$ I = 10.17 mA

In these conditions of moderate tune depression ($\eta > 0.5$)
(what do we see when $\eta < 0.5$???)

3D simulations have always shown that high order coupling resonances :

$$\alpha = \sigma_y / \sigma_x = 1/3 \quad (8^{th} \text{ order})$$
$$\alpha = \sigma_y / \sigma_x = 1/2 \quad (6^{th} \text{ order})$$
$$\alpha = \sigma_y / \sigma_x = 2 \quad (6^{th} \text{ order})$$
$$\alpha = \sigma_y / \sigma_x = 3 \quad (8^{th} \text{ order})$$

…

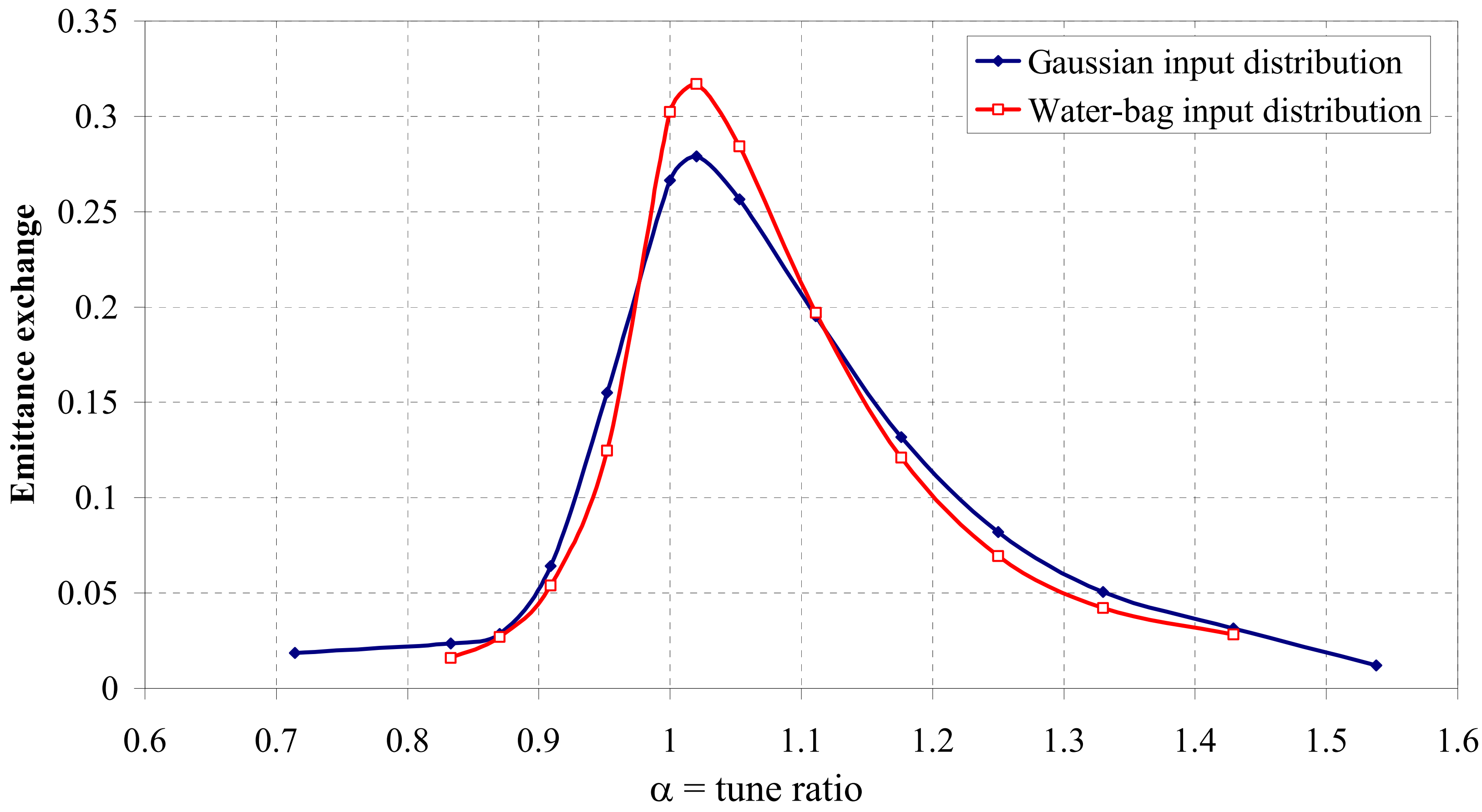
have negligible effects for a large range of energy ratio

Coupling effects "visible" only for the 4^{th} order coupling resonance

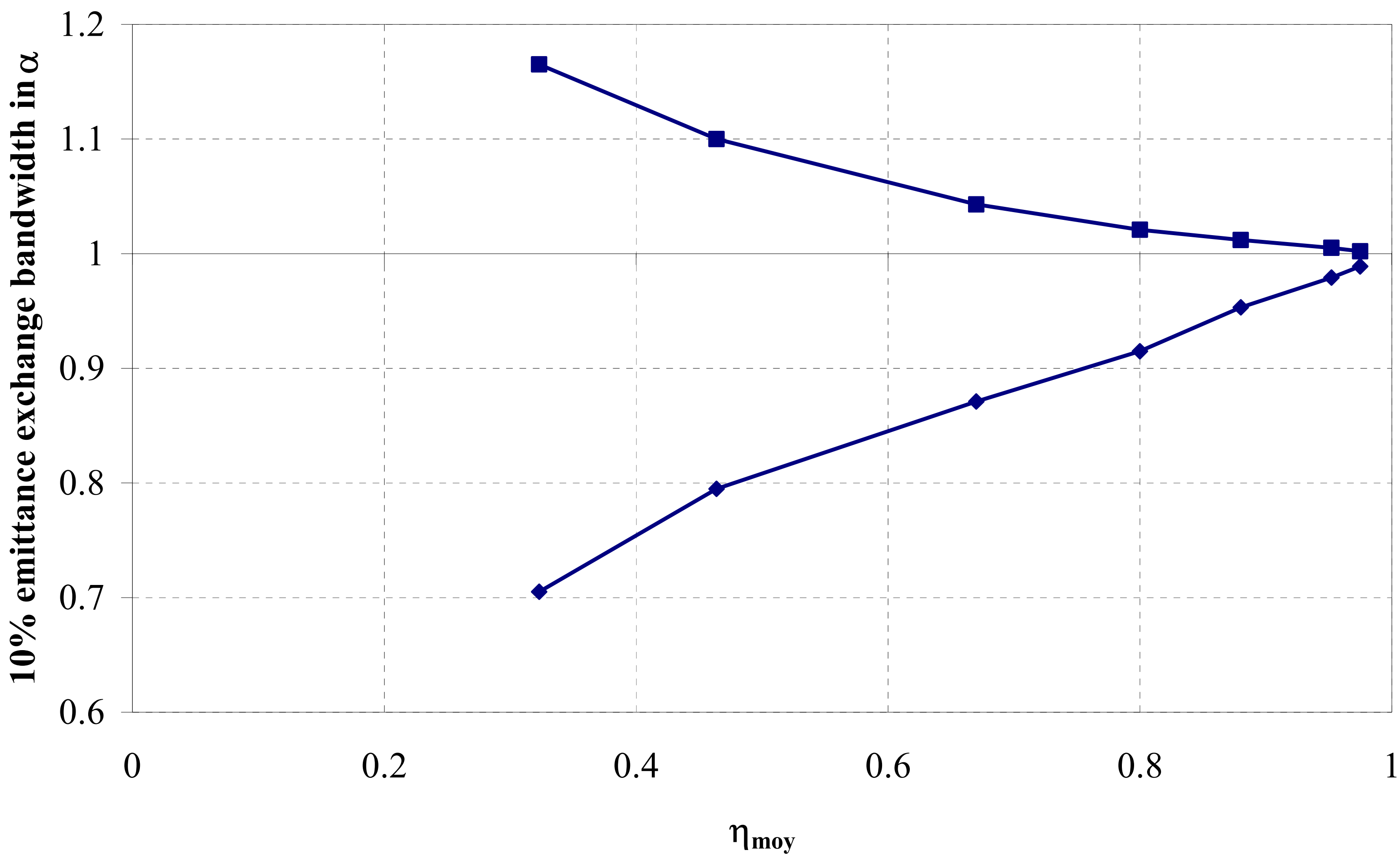
V coefficients	Excited coupling resonances	$\alpha = \sigma_y / \sigma_x$	
$k_{22}\, x^2\, y^2$	$2\,\sigma_x - 2\,\sigma_y = 0$	$\alpha = 1$	**4^{th} order**

Study of the $2\,\sigma_x - 2\,\sigma_y = 0$ 4th order coupling resonance

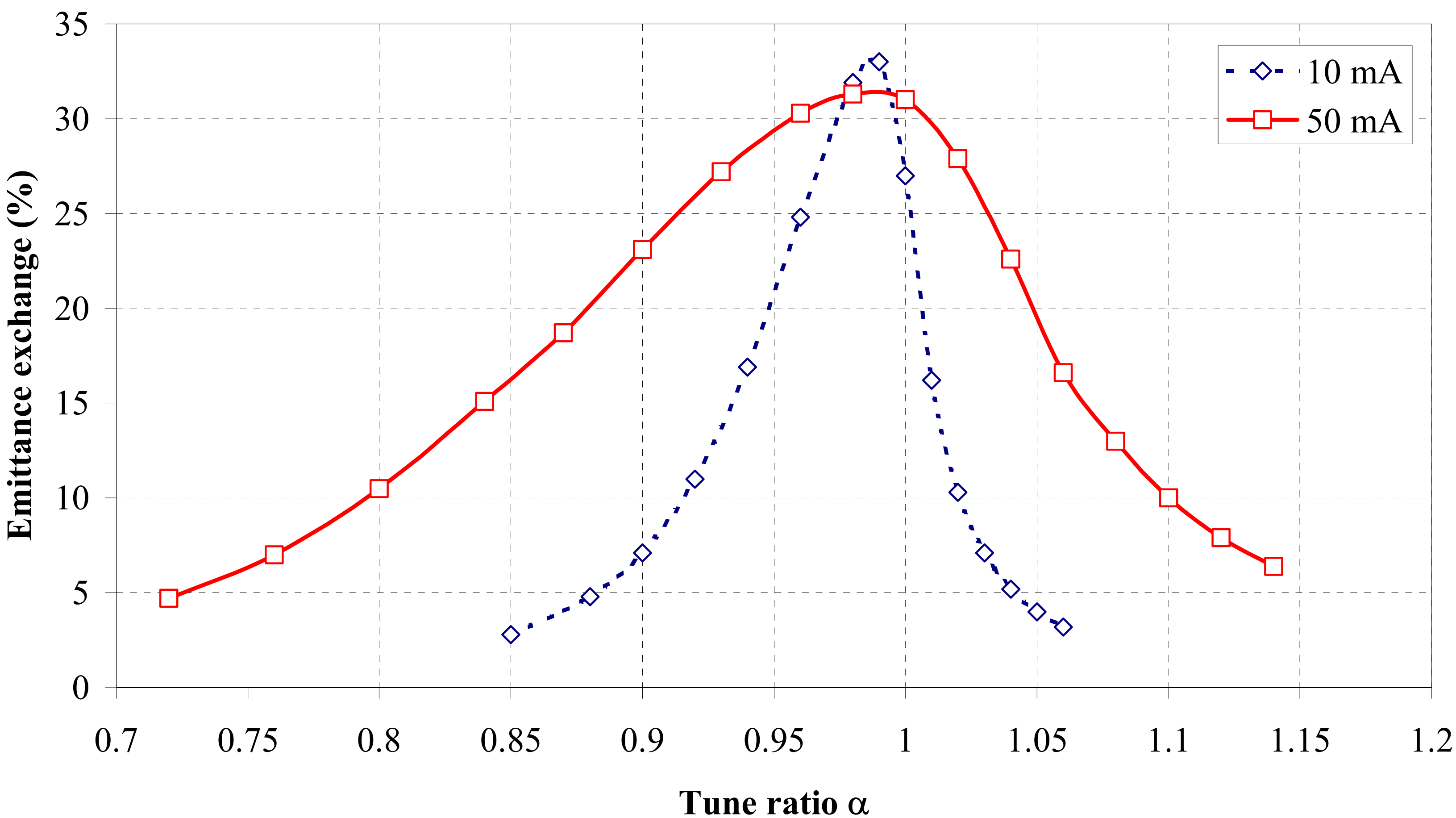
Many 3D PIC simulations (N. Pichoff, D. Uriot)



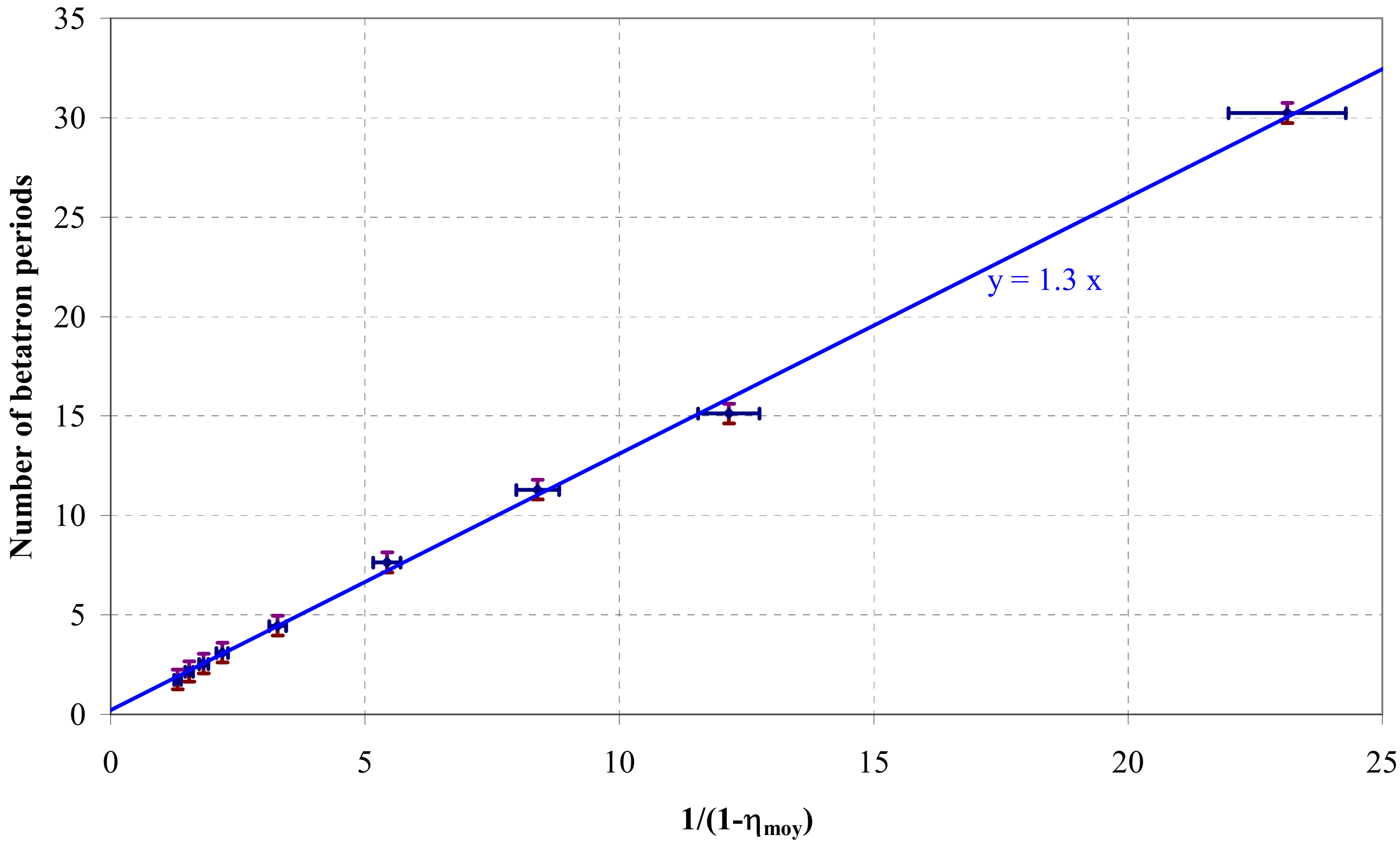
Emittance transfers weakly dependant of the type of density distribution



The width of the coupling resonance increases as space charge increases ($\eta \downarrow$)



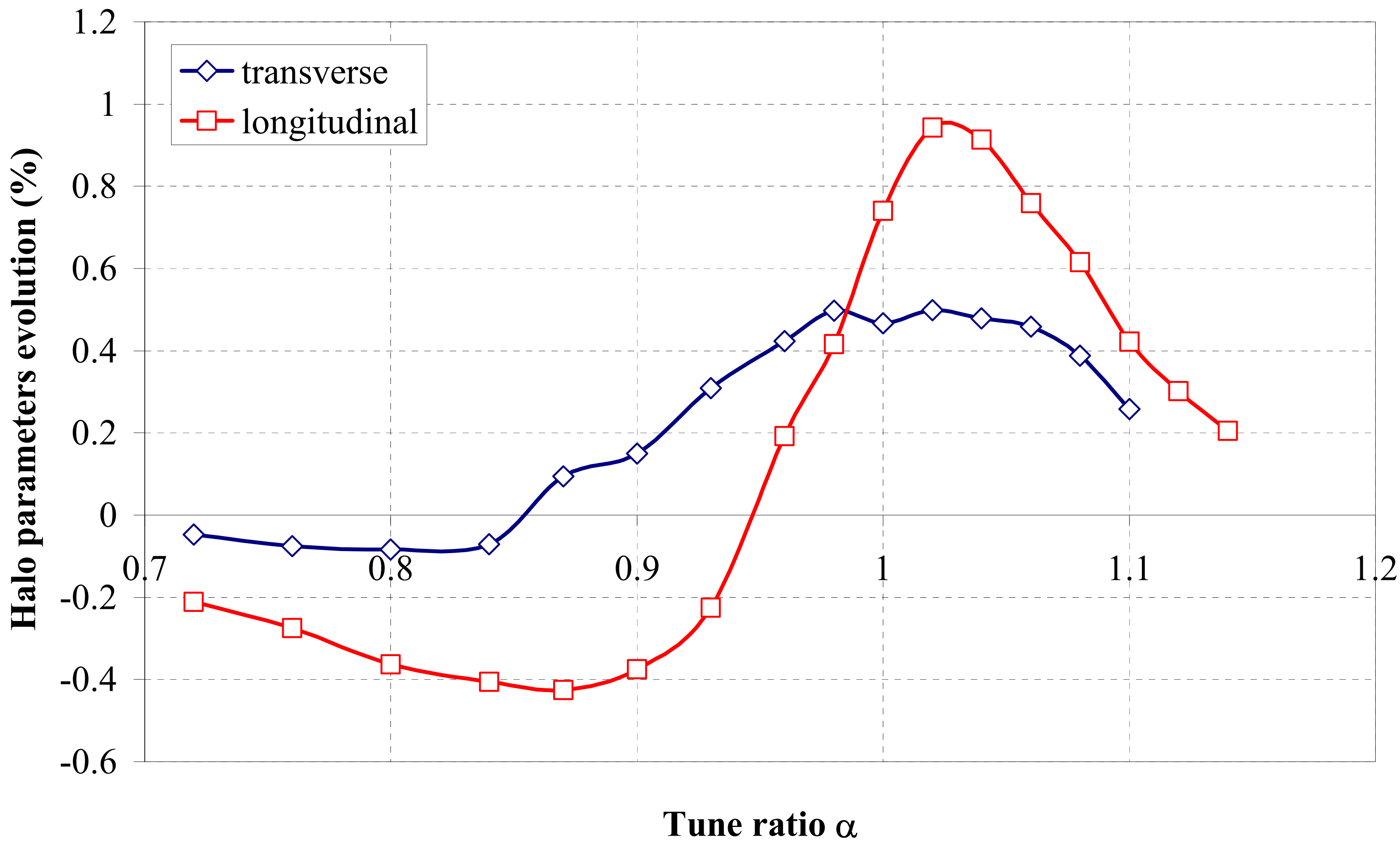
On the resonance, the amplitude of the emittance exchange is weakly dependant of space charge



The length (time) to reach the equilibrium is function of space charge

η	1	0.9	0.8	0.7	0.5	0.2	0.1
$1/(1-\eta)$	∞	10	5.0	3.3	2.0	1.25	1.11

~ 7 βtron periods for $\eta = 0.8$ ~ 2.5 βtron periods for $\eta = 0.5$



Particles of the core can be transferred to the halo
Particle of the halo can be transferred into the core
Behavior function of the working point and coupling resonance respective positions