

Spin Orbital Function Formalism and ASPIRRIN Code

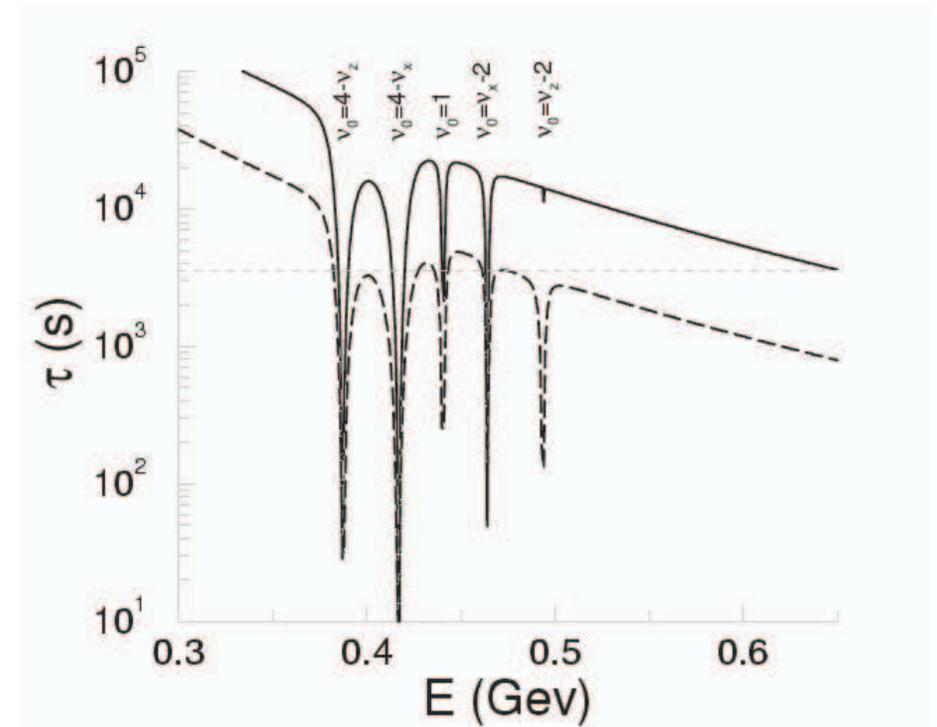
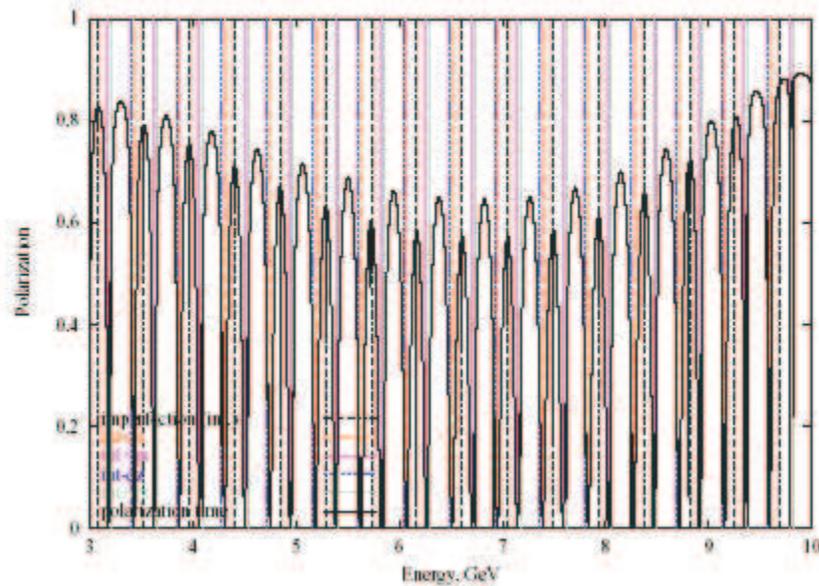
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ASPIRRIN code

- Analysis of **SPI**n Resonances in **RIN**gs
- Polarization calculation in accelerators with complex configuration of magnet field. To calculate snakes and rotators which use solenoidal and horizontal fields.
- Calculation of the equilibrium polarization and the depolarization time in electron(positron) rings.
- Restricted to the calculation of first-order (linear) spin resonances. (On the same level as SLIM/SLICK codes).
- Calculation of Spin Orbital functions, which can be used for resonance calculation and analysis.

ASPIRRIN Calculations

Vertical “dog-leg” scheme
No spin transparency



Calculations for eRHIC

ASPIRRIN calculation
of polarization time
for VEPP-2M

Spin Motion

$$\frac{d\mathbf{S}}{d\theta} = (\mathbf{W}_0 + \mathbf{w}) \times \mathbf{S}$$

Precession on the ideal (reference) orbit

$$\hat{\mathbf{n}}_0(\theta + 2\pi) = \hat{\mathbf{n}}_0(\theta)$$

$$\nu = \phi/(2\pi) \quad (\text{Spin tune})$$

$$(\hat{\mathbf{n}}_0, \hat{\eta}_1, \hat{\eta}_2)$$

(Orthogonal spin basis)

Precession due to betatron/synchrotron motion and closed orbit error

$$\hat{\mathbf{n}}(x, p_x, z, p_z, \sigma, p_\sigma, \theta + 2\pi) = \hat{\mathbf{n}}(x, p_x, z, p_z, \sigma, p_\sigma, \theta)$$

Vector function of the point in the 6-D phase space

Observable polarization: $P(t) \langle \hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0 \rangle(\theta)$

Critical for electrons

Critical for protons

Polarization evolution

Synchrotron radiation introduces both polarizing and depolarizing effects which lead to the equilibrium polarization:

$$P(t) = (P_0 - P_{eq}) e^{-t/\tau} + P_{eq}$$

Derbenev-Kondratenko:
(1973)

$$\mathbf{d} = \frac{\partial \hat{\mathbf{n}}}{\partial p_\sigma}$$

defines strength of depolarization
and kinetic polarization

$$P_{eq} = -\frac{8}{5\sqrt{3}} \frac{\alpha_-}{\alpha_+}$$

$$\tau^{-1} = \frac{5\sqrt{3} \hbar r_0}{8 m} \gamma^5 \alpha_+$$

$$\alpha_- = \frac{1}{R_0^3} \left\langle \frac{\hat{\mathbf{b}}}{|K|^3} (\hat{\mathbf{n}} - \mathbf{d}) \right\rangle$$

$$\alpha_+ = \frac{1}{R_0^3} \left\langle \frac{1}{|K|^3} \left[1 - \frac{2}{9} (\hat{\mathbf{n}} \hat{\mathbf{v}})^2 + \frac{11}{18} |\mathbf{d}|^2 \right] \right\rangle$$

Linear approximation

Spin vector has two independent degree of freedom and can be characterized by a complex variable C :

$$(|C| = |S_{\perp}|)$$

$$\hat{\mathbf{n}} = \sqrt{1 - |C|^2} \hat{\mathbf{n}}_0 + \text{Re}(iC \hat{\eta}^*)$$

$$\hat{\eta} = \hat{\eta}_1 - i\hat{\eta}_2$$

The spin equation becomes:

$$C' = \sqrt{1 - |C|^2} w_{\perp} - iw_{\parallel} C$$

$$|C| \ll 1$$

First order in spin, first order in orbit:

$$C' = w_{\perp} = W^T X + w_0$$

Sextupole and higher order field are ignored

ASPIRRIN Algorithm

Look for a solution in the form:

$$C = (1 + \nu_0)(z' \eta_x - x' \eta_z) + f_0 + F^T S X$$

6D vector

$$(x, p_x, z, p_z, \sigma, p_\sigma)$$

Symmetric hamiltonian matrix

$$S = \begin{pmatrix} S_0 & 0 & 0 \\ 0 & S_0 & 0 \\ 0 & 0 & S_0 \end{pmatrix}$$

$$S_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Then equations are:

$$F' = SHF + P$$

$$f'_0 = \Delta K_z F_1 - \Delta K_x F_3 + (1 + a) \Delta K_y \eta_y$$

Transfer Matrix

Linear orbital motion

$$X' = SHX + Q$$

Linear spin motion

$$F' = SHF + P$$

The same

linear part

Only in bends
and solenoids

$$X = MX + R$$

$$F = MF + V$$

provides the same transfer matrix

M

One turn transformation

$$F(2\pi) = M_{rev}F(0) + Y_{rev}$$

Orbital one turn
transfer matrix

$$F_i(\theta + 2\pi) = e^{i2\pi\nu} F_i(\theta)$$

Periodicity condition

$$F(0) = (I \cdot e^{i2\pi\nu} - M_{rev})^{-1} Y_{rev}$$

$$\nu = m \pm \nu_i$$

First order resonances (linear resonances)

Betatron and synchrotron tunes

Spin-Orbital functions

✦ F_i functions are called spin-orbital functions

✦ F_5 is related with spin-orbital vector: $\mathbf{d} = \text{Re}(iF_5\hat{\eta}^*)$

$|\mathbf{d}| = |F_5|$ periodical function of ring azimuth

✦ F_1 and F_3 can be used to describe depolarizing effects related with change of particle transverse momentum (target scattering, beam-beam)

✦ Spin resonances calculations

Examples of resonance strength calculations

$$|w_k| = \frac{1}{2\pi} \left| \int_0^{2\pi} (\Delta K_x F_1 - \Delta K_x F_3 + (1+a)\Delta K_y \eta_y) d\theta \right|$$

Imperfection resonance
due to field errors

$$|w_{\nu_k}| = \frac{1}{2\pi} |A_x| \left| \int_0^{2\pi} \kappa F_3(\nu = \nu_k) f_x^0 d\theta \right|$$

resonances caused by weak coupling

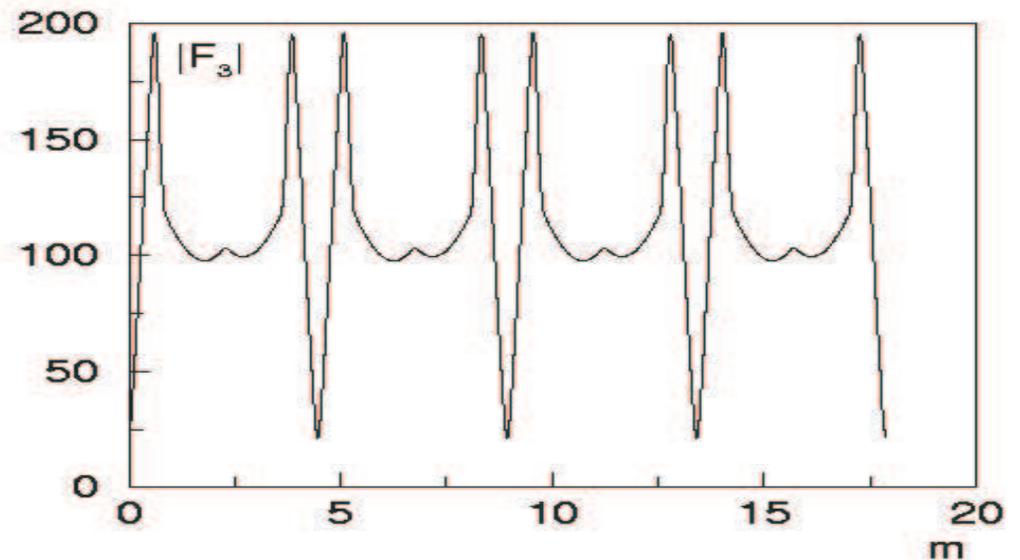
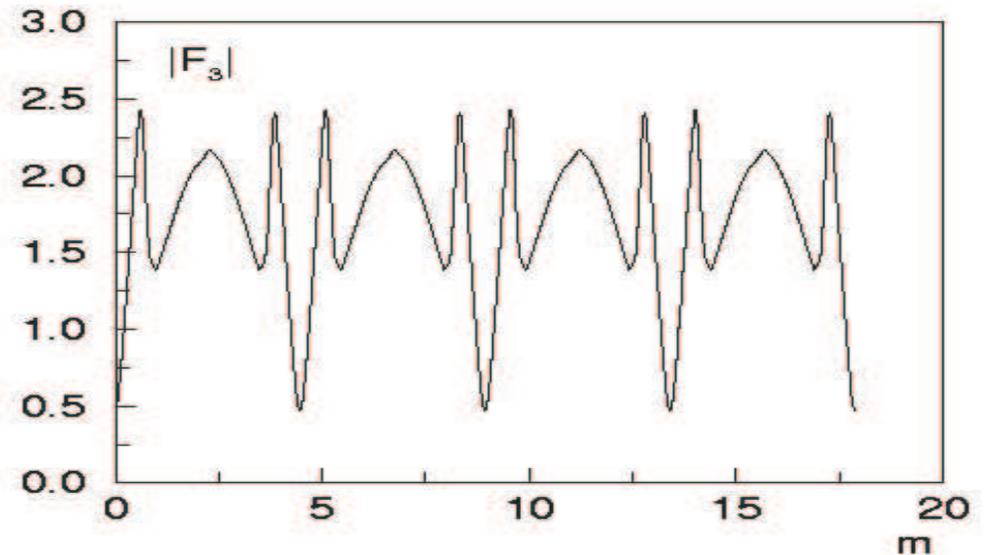
$$|w_k| = \frac{|p_\sigma|}{2\pi} \left| \int_0^{2\pi} \kappa \psi_x F_3 d\theta \right|$$

$$|w_{km}| = \frac{\nu_\sigma |\tilde{w}_k|}{\nu_0} m J_m \left(\frac{\nu_0 \Delta_p}{\nu_\sigma} \right)$$

synchrotron sidebands

Standard accelerator

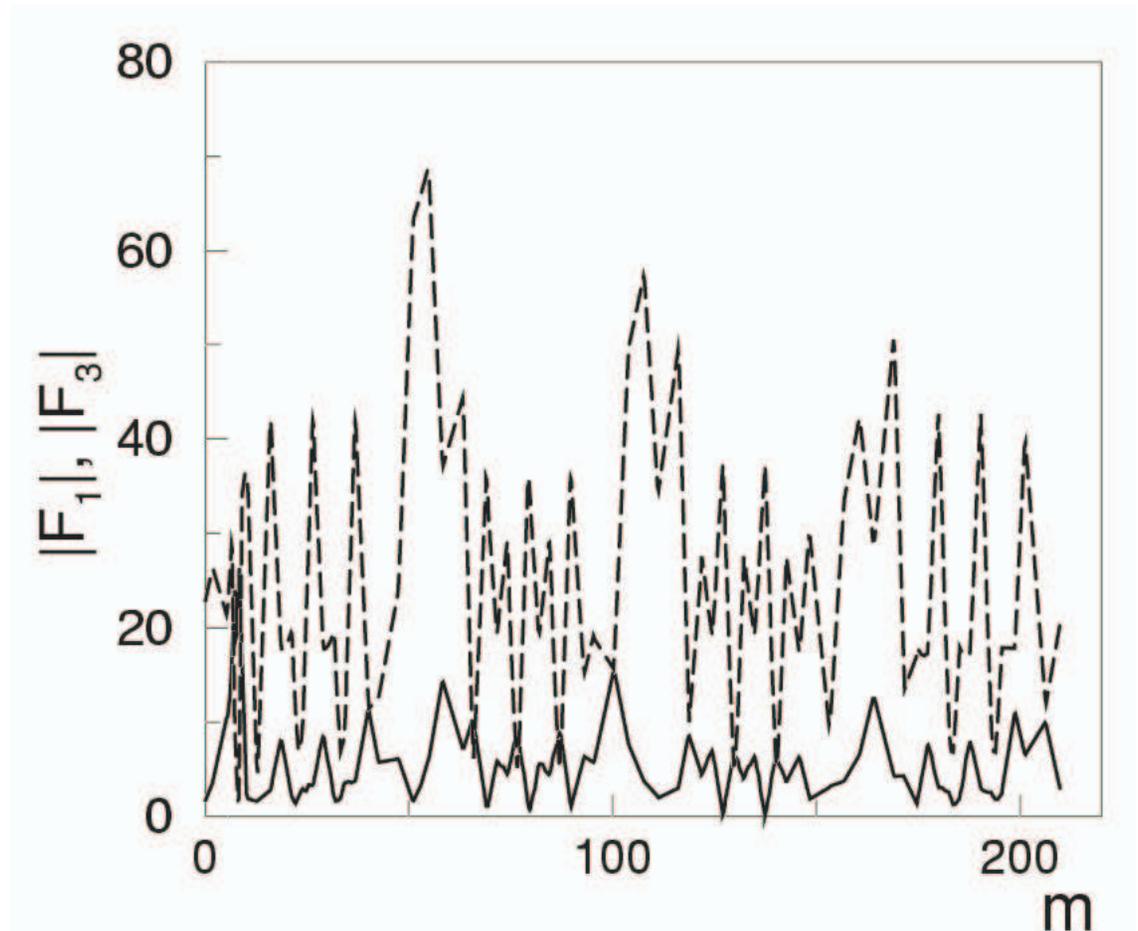
- ★ Only vertical bending field; no solenoids
- ★ Spin motion only coupled with vertical orbital motion. Only F_3 , F_4 are nonzero.
- ★ F_3 , F_4 increasing near spin resonances with vertical betatron tune



Accelerator with a snake

AmPS with unmatched snake

There is large F1 component around the ring (dashed)



F₅ function

The equation for F₅ function (4D+2 approximation):

$$F'_5 = K_z(\nu_0\eta_z - F_1) + K_x(\nu_0\eta_x + F_3) - K_y\eta_y$$

bending magnets



solenoids



Changes are only in regions with a field on the reference orbit

In ideal accelerator (vertical bends only, no coupling error)

F₅ is 0

F₅ general solution

$$F_5 = F_{5\gamma} + F_{5\beta} \quad \leftarrow \text{divided into two components}$$

$$F_{5\gamma}(\theta) = \frac{1}{e^{i2\pi\nu} - 1} \int_{\theta}^{\theta+2\pi} (V^T(\theta')\Psi(\theta') + V_{\delta}(\theta')) d\theta' \quad \leftarrow \text{resonant with integer}$$

$$F_{5\beta}(\theta) = - \sum_{j,k,n=1}^4 \frac{\mathcal{F}_{jk}^{-1}(\theta)\Psi_k(\theta)}{e^{i2\pi(\nu+\nu_j)} - 1} \int_{\theta}^{\theta+2\pi} V_n^T(\theta')\mathcal{F}_{nj}(\theta') d\theta' \quad \leftarrow \text{resonant with orbital tune}$$

$$\tilde{X} = \mathcal{F}\mathcal{A} + \Psi p_{\sigma}$$

orbital motion solution

$$\begin{aligned} V_1 &= K'_x \eta_y + \nu_0 \frac{K_y^2}{2} \eta_z \\ V_2 &= (\nu_0^2 K_x \eta_y - \nu_0 K_y \eta_x) \\ V_3 &= K'_z \eta_y - \nu_0 \frac{K_y^2}{2} \eta_x \\ V_4 &= (\nu_0^2 K_z \eta_y - \nu_0 K_y \eta_z) \end{aligned}$$

$$V_{\delta} = \nu_0 K_x \eta_x + \nu_0 K_z \eta_z - K_y \eta_y$$

Example of spin matching conditions

- Zero dispersion functions inside the snake insertion
- No betatron coupling outside the insertion

Two solenoids snake:

$$F_{5\beta} = -\frac{\nu_0\pi}{4 \cos(\pi\nu_x)} \left[\cos(\pi\nu_0) \operatorname{Im}(e^{i\pi\nu_x} J(\theta) G_{Ix}^*) + i \operatorname{Im}(e^{i\pi\nu_x} G_{Iz}^* J(\theta)) \right]$$

$$G_{Ix,z} = f'_{Ix,z(out)} - f'_{Ix,z(in)}$$

Exit of second solenoid

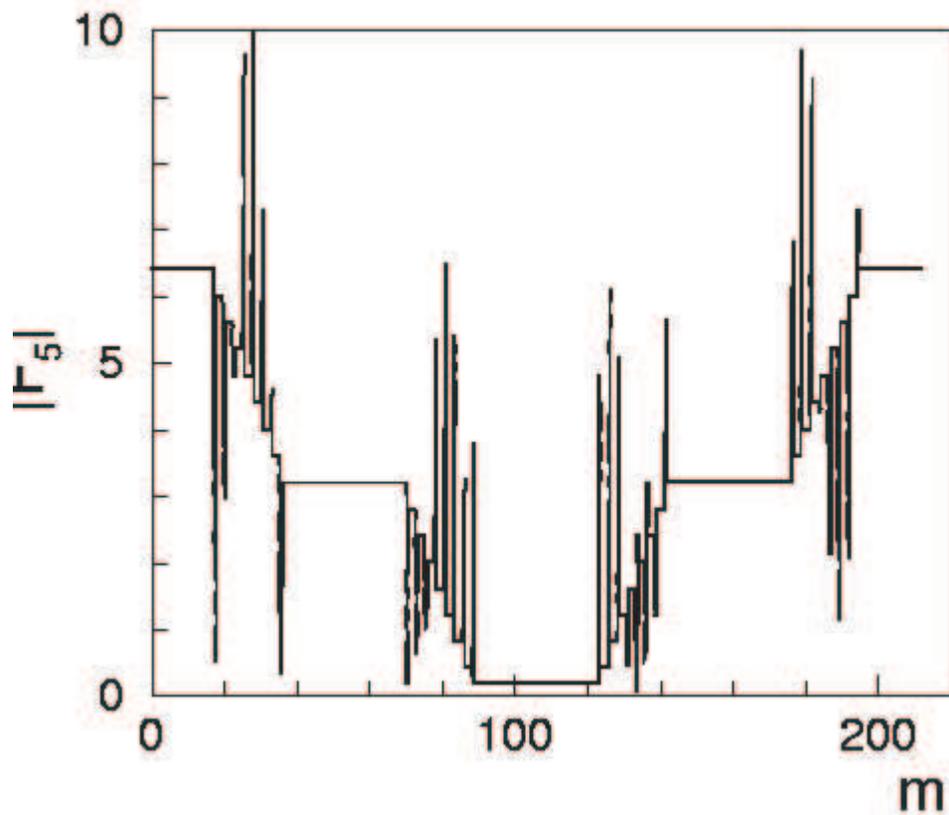
Entrance of first solenoid

$$J(\theta) = f_{Ix}\psi'_x - f'_{Ix}\psi_x$$

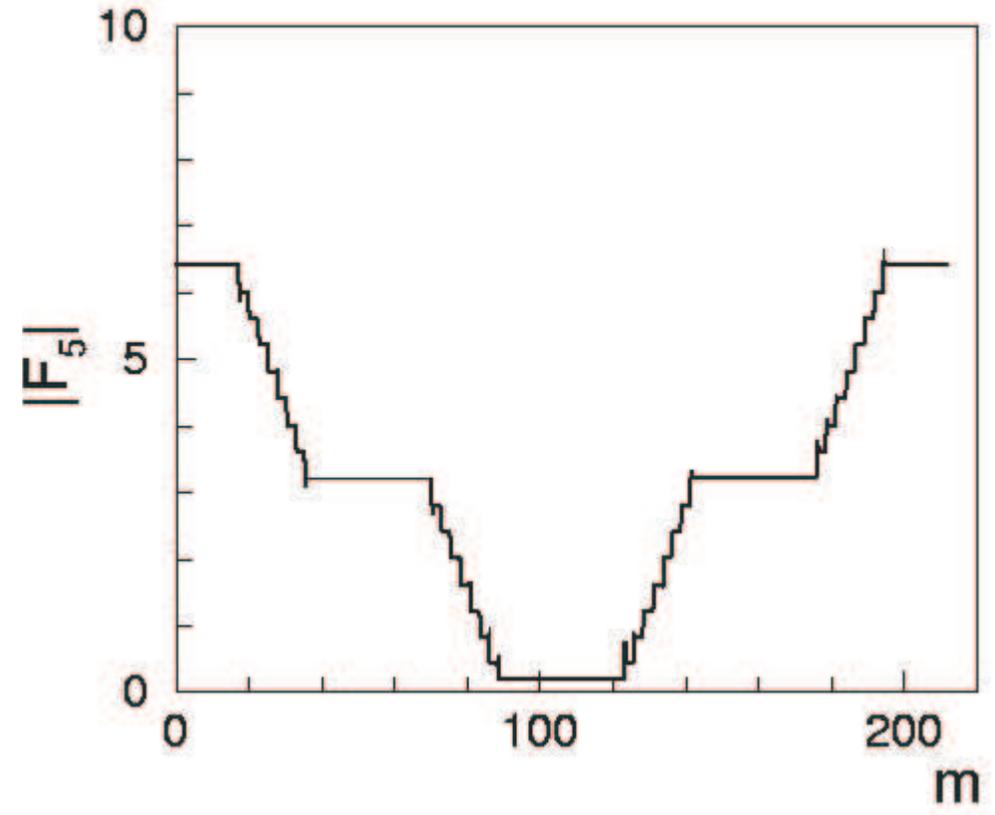
$$f'_{Ix(out)} = f'_{Ix(in)}, \quad f'_{Iz(out)} = f'_{Iz(in)}$$

F_5 function

Unmatched case



Matched case (reversed gradients)



ASPIRRIN calculation for AmPS

Summary

- Spin Orbital formalism and ASPIRRIN code were designed for psolarization calculation in accelerators with complex configuration of magnet field at the first-order level.
- Calculation of the equilibrium polarization and the depolarization
- Analysis of spin matching/transparency conditions.
- Calculation of Spin Orbital functions, which can be used for resonance calculation and analysis.
- Currently no magnet/alignment errors capability in the code input file.